

Secondary School Students' Errors and Misconceptions in Learning Algebra

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ABSTRACT

The aim of the study is to develop an understanding of the kinds and sources errors and misconceptions that characterise students' learning of school algebra. Systematic random sampling was used to draw sixty-five participants from a population of two hundred and twenty-three form three students. A cross sectional survey design was employed to collect data using written tests, a structured questionnaire and interviewing of the students from one high school in Zimbabwe. Content analysis technique was applied to textual data from three sources in order to determine the types of errors and misconceptions. The main findings are that both procedural and conceptual errors were prevalent that errors and misconceptions can be explained in terms of the students' limited understanding of the nature of algebra; in particular their fragile grasp of the notion of a variable. Sources of misconceptions could be explained in terms of the abstract nature of algebra Mathematics educators should embrace errors and misconceptions in their teaching and should not regard them as obstacles to learning but rather engage with them for better understanding of algebraic concepts by students. Future studies can be carried on systematic errors as one of the ways of improving students' understanding school mathematics.

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1. INTRODUCTION

1.1. Motivation and Context of the Study

This study was motivated by persistent poor results in Mathematics. There is widespread interest among nations in improving the levels of mathematics achievement in schools. Mathematics commands an enviable position in our everyday life. The importance of learning algebra is widely acknowledged. Algebra is used ever widely in scientific disciplines like engineering. It still commands a central role in advanced mathematics. Tecla [1] suggests that in our march towards scientific and technological advancement we need nothing short of good performance in Mathematics at all levels of schooling. Strides towards the attainment of high levels of mathematics achievement among learners are hampered by errors and misconceptions secondary school students often make and misconceptions in learning algebra. The students lack deep understanding of this domain. A superficial knowledge of algebra may affect understanding other mathematics and scientific disciplines

Gillian [2] affirms that abstract algebra is important in the education of mathematically trained person. However, Witzel [3] found that despite the significance of algebra in school mathematics curricula, many students find it difficult to comprehend. Many attempts to prepare students for algebra have not yielded greater achievement. Further, the study of abstract algebra, until recently has placed significant emphasis on

college algebra and little emphasis has been given to school algebra which is a crucial requisite knowledge for learning of college and higher forms of algebra. Secondary school students continue to struggle with algebraic concepts and skills and many discontinue their study of advanced mathematics because of their lack of success in algebra as reported by Greens [4]. Gatawa [5] it is a challenge to help students overcome their frustrations, but necessary effort because of the importance of mathematics. Hence, there is need for critical attention to unearth students' conceptualisation of algebra and make an in-depth analysis of kinds and sources errors and misconceptions to promote deep learning and engagement.

1.2. Statement of the Problem

The persistent high failure rates in school mathematics in Zimbabwean schools have been a major concern for a long time and needs critical attention. There is lack of deep understanding of secondary school students' persistent struggles with algebra and types and sources of errors and misconceptions made by the students when learning algebra. To address these concerns the researchers raised and addressed the following research questions.

1.3. Research questions

- (i). How do secondary school students conceive algebra?
- (ii). What kinds of errors and misconceptions do students experience in learning school algebra?
- (iii). What are the possible sources of errors and misconceptions in the domain of school algebra?

1.4. Theoretical Bases

1.4.1. Theoretical Framework

The guiding philosophy that was considered to be compatible for this study of secondary school students' thinking about algebraic concepts is constructivism. Constructivism as an underpinning learning theory was used as lenses to view and illuminate secondary school students' conceptualizations of algebraic concepts. The ideas of constructivism are omnipresent in modern pedagogical theory and practice. Constructivism asserts that concepts are shaped in the learning process during the sense making process when new information filters through the student's mental schemata. Ndemo [6] write that the process of information filtering depends on the student's met-before, which is a collection of prior knowledge, beliefs, prejudices, preconceptions and misconceptions. The fundamental ideas are that students come to the classroom with different experiences from their social lives. In the teaching-learning encounters new knowledge can be constructed by sharing these experiences. Hence, mathematics learning is a constructive process and mathematics knowledge is constructed from related knowledge the student has acquired. During the knowledge construction process learners can actively construct and reconstruct their own experiences as asserted by Prince [7]. The constructivist school of thought holds that students' efforts to construct knowledge may involve explaining their reasoning— a crucial component of the learning of algebraic concepts that informed the construction of the research instruments such as the written tests.

According to Polya [8] students' involvement is essential for improving performance. Mtetwa [9] says by involvement, it means how much time, energy, effort students devote to the learning process. Unameh [10] posits that the more time and effort students invest in the learning process, the more intensely they engage in their own education, the greater will be their growth in mathematics knowledge, achievement, their satisfaction with their educational experiences and their persistence in school and more likely they are to continue their learning. Students actively construct their individual mathematical worlds by reorganising their experience. The students' reorganised experiences form a personal mathematical structure that is more powerful, more complex and more abstract than it was prior to the reorganisation.

1.4.2. Mathematical underpinnings of algebra

There are many conceptions regarding the nature of algebra in literature and in the current secondary school curricula. The structural features of algebra in terms of variables, symbolic expressions, algebraic equations, functions and inequalities are connected together to form a broader conception of algebra. Gunawaden [11] writes that a variable is hard to define because its definition largely depends on the context. Usiskin [12] describes the four possible meanings of a variable based on the fundamental conceptions of algebra.

The first conception of algebra is that algebra is generalised arithmetic. In this view algebra is seen as the study of structures, relations, equality and substituting numbers. Usiskin [12] suggests that in this sense algebra has been transformed into many forms of mathematics like analytic geometry and calculus because of the power of algebra as generalised arithmetic whereby the focus is on variables and the relationships among these changing quantities. In this conception, a variable is a pattern generaliser. For example, the arithmetic

expressions like $-3 \times 2 = -6$ could be generalised to give a property like $-x \times y = -xy$. The commutative property of addition $3 + 2 = 2 + 3$ could be generalised to $x + y = y + x$ in algebra.

The second conception is one in which algebra is seen as the study of structures whereby the notion of a variable is considered as an arbitrary element of algebraic reasoning. Within Usiskin's [12] second category of algebraic understanding a variable is seen as an arbitrary object in a structure related by certain properties. This is the widely held view found in abstract algebra. Usiskin [12] proposes a third category of algebraic understanding that accepts algebra as the study of relationships. An exemplification of this conception of algebra can be discerned from the way the concept of a function is understood. Precisely, a function is a rule that associates with each in the domain a unique element in another set called the range of the function.

In the fourth conception, Usiskin [12] view algebra as procedures for solving problems and a variable is viewed as an unknown, which is clearly related with equations. An equation has expressions combined by an equal sign. To solve an equation correctly, the student must know the rules or procedures of simplifying algebraic expressions. This conception suggests that algebra is the study of procedures for solving problems. In this conception there is need to apply both heuristic and procedural techniques to generate a solution for the unknown. For example, the following problem can be posed to secondary school student. *When 3 is added to 5 times a certain number the result is 40. Find the number.* The solution process can involve translating the narrative form into algebraic form such as $5x + 3 = 40$, that would lead to the solution $x = 7.4$. Another example of cases typical of Usiskin's [12] fourth category of algebraic reasoning can be seen in students' attempts to solve inequalities. An inequality is also called an inequation whereby order properties of the ordered field of real numbers are employed in solving problems on inequalities.

For the purpose of this study, ideas drawn from Usiskin's first and fourth categories of algebraic understanding were employed in investigating secondary school students' conceptions of algebra as well as attempting to determine the kinds of errors and misconceptions and their possible sources during the learners' encounters with algebraic concepts. The justification is that the first level of algebraic reasoning was considered suitable for the secondary school level of learning whereby algebra is seen as generalised arithmetic and hence algebraic reasoning demanded at this level would be within the conceptual reach of the secondary school learner who is at the transitional phase from arithmetic of numbers in the primary school curriculum to the secondary school phase where generalised arithmetic proposed by Usiskin [12] would be introduced. The central idea drawn from Usiskin's fourth category of algebraic is the aspect of problem solving that allowed the researchers to determine kinds of misconceptions and errors from students' solution attempts of assigned algebraic tasks

2. RESEARCH METHOD

2.1. Research design

The main goal of the study was to characterise secondary school students' conceptions of algebra, kinds and sources of students' errors and misconception in this domain. Informed by an understanding that an individual's conception of mathematical concepts is contextual and can be revealed through in-depth investigation, the researchers employed a cross-sectional research design to study the kinds of misconceptions held by secondary school students in the Zimbabwean context. The cross-sectional survey was considered suitable because the design helped to tease out students' thinking process, errors and misconceptions in algebra. Following Descombe [13], the design provided us with an opportunity to study the secondary school students in their natural setting. A qualitative research paradigm, informed by rationale of developing an understanding of the nature of errors and misconceptions held by the students was adopted for this study. Usually, qualitative research questions start with how and what questions. While quantitative researchers attempt to establish universal contexts-free generalisations, our study context compelled us to develop context-bound generalisations about students' kinds of algebraic thinking and misconceptions made when students attempted to solve written tasks involving algebraic concepts that is consistent with McMillan's [14] suggestions of in-depth qualitative studies can be conducted.

2.2. Population and sampling

Systematic random sampling was used to draw sixty-five participants from a population of two hundred and twenty-three form three students at one boarding secondary school in Zimbabwe. The school was selected because the students had the necessary background study of algebra. All participants passed had primary school mathematics. The participants were adolescents in the 15 – 17 age range. English language was the medium of instruction for school mathematics learning. We selected Form 3 students because it is at secondary school level of learning that students are expected to develop a strong foundation for understanding the algebraic concepts that are relevant and necessary for studying mathematics at higher

levels. The first of this article had a prolonged engagement with the participants for teaching term that lasted for 12 weeks. Prolonged interaction with the students allowed the researchers to generate data from multiple sources for triangulation purposes.

2.3. Data collection procedures

Data gathering was done by the first author who was resident at the research site. By being one of the mathematics teachers at the school, the main author did not experience entry related challenges. Each student participant signed a consent form after being assured that participation was voluntary and that participants were free to withdraw from the study without fearing for retribution from the researcher who was the informants' mathematics teacher. Written tests, in-depth interviewing guide, a structured questionnaire were used as data gathering instruments.

2.3.1 Written test

The first author administered the written tasks to sixty-five participants. The test contained twenty-five items. Participants were required to define the word a variable generate examples and non-examples of a variable. Tasks involved formulating algebraic expressions from word problems, making some quantitative comparisons, solving a system of equations in two variables, solving equations and inequalities. Furthermore, tasks included assessed secondary school students' abilities to identify patterns or relationships and represent them algebraically. Written tasks included items that focused on students' justification skills whereby students were required to justify their answers and algebraic methods used to solve problems.

2.3.2 Questionnaire

The participants completed a structured questionnaire. The short questions required participant to provide a true or false response. The questionnaire provided background information about students' prior knowledge and attitude towards algebra and personal demographic data. Students were also required to indicate their grade seven mathematics result, their gender, age and state whether day scholar or boarder. These data were critical for interpretation purposes of the kinds of thoughts displayed when students engaged with the tasks.

2.3.3 Student interviews

The researcher conducted one-to-one interviews to gain insights into the students' understanding of algebra, procedures solving tasks in algebra and identify error patterns and misconceptions. Five interviewees were selected by thoroughly examining the answers in the test. The students who displayed serious, pernicious errors and misconceptions were selected for interviews. The interview schedule involved the student reading the question, comprehension, strategy selection, processing (working out questions), explaining procedures, encoding, consolidating and verification. The students explained their answers to elaborate errors and misconceptions. Interviewing allowed for the exploration of students' algebraic reasoning abilities. Students articulated their thoughts and verbalised their actions to ensure insights into their thinking processes. During such mental operations, insufficiencies were spotted. All of these interviews were audio taped and transcribed.

2.4. Data presentation and analysis procedures

Since the aim of the study as to explore students' understanding of algebra and identify students' errors underlying misconceptions blank and non-informative responses were left out in the presentation and analysis of data. Rubric of error categories were tabulated as per conceptual area for easy interpretation. Literature on conceptions of algebra, types and sources of errors and misconceptions guided in generating themes of error categories. Wrong responses were compiled and grouped according to type of error or misconception. The research was open ended to include other categories of errors and misconceptions. The researcher analytically identified errors for other categories not in literature. Data was presented as per research question.

Content analysis of students' written tests, questionnaire responses and interview transcriptions constituted data analysis. Data analysis and interpretation entailed critical examination rather than mere description of students' responses in the written test, questionnaire and interview transcripts. It is an analytical method used in qualitative research to gain understanding of trends and patterns that emerge from data. Following Daymon [15] content analysis we employed allowed us to discover patterns and categories of students' errors and misconceptions from students' workings.

3. RESULTS AND DISCUSSION

This section presents research findings per research question. The results were from content analysis of textual data from structured questionnaire, two written tests and five students' interviews. Focus was mainly on students' conceptions of algebra, the kinds of errors, misconceptions and their origins. The twenty-five test items were classified into one of the five conceptual areas; variables, expressions, functions, equations and inequalities. The errors and possible misconceptions in each question item were noted and put into various categories. However, this classification was non-exhaustive as there were some non-compliant cases.

3.1. Research question one results: *How do secondary school students conceive algebra?*

Students' responses revealed that students were aware that a letter can represent any number in algebra. A typical example of this dominant conception of algebra among secondary school student informant is now presented.

1. What is a variable? Explain with an example.

Answer: *An algebraic letter which appears on different algebras a or b*

The students' understanding of a variable was consistent with the formal definition of a variable as pattern generalise Usiskin [12]. The students understood that in algebra letters are used as arbitrary objects in algebraic expressions. While such responses revealed that students appreciated the role played by use of letters in algebra, students demonstrated a weak understanding of the connection between arithmetic and algebra. For instance, they were not aware that rules of precedence are also applicable to algebra. This finding confirms Greens' [4] observation that students still struggle with algebraic concepts. Some students did not realise that understanding arithmetic is key to understanding of algebra.

3.2. Research question 2 results: *What kinds of errors and misconceptions do students experience in learning school algebra?*

Students' written tests manifested various taxonomies of errors and misconceptions. The broad categories of errors noted were conceptual errors, computational and procedural errors. Conceptual errors are caused by students' inadequate knowledge of concepts. Computational errors are calculation errors. Procedural errors are a result of a wrong or incorrect method in the process of solving a problem.

3.2.1. Students' errors and misconceptions on variables

Students misinterpreted a variable as a label or as a thing or a verb. They failed to perceive the variable as the number of a thing. Schoenfeld [16] assert that understanding the concept of a variable is central for transition from arithmetic to algebra. When asked to generate examples and non-examples of a variable, students considered words such as sweets or cents as symbols representing variables. However, these answers were incorrect in the context of the given question because variable or a symbol was supposed to be used to represent such letters such as x for sweets and y for the cost of the sweets. Students' conceptions of the notion of a variable were also explored posing the question: *Apples cost a cents and bananas cost b cents. If 3 apples and 2 bananas are sold, what does $3a + 2b$ represent?* Students' conceptions of a variable shown in Table 1.

Table 1. Students' conceptions of a variable

Expected response	Examples of elicited responses
The total cost of 3 apples and 2 bananas	$\times a + 2 \times b$ 3cents and 2cents $5ab$ Expression
The total cost of 3 apples and 2 bananas	Price of apples and bananas 3 apples and 2 bananas

Students displayed lack of understanding of the unitary concept when dealing with variables. This is a basic arithmetic concept. From Table 1, students wrote $5ab$ which was a serious misconception of adding unlike terms. In addition to the incorrect addition of unlike terms, the students regarded a as the label for apples and b as the label for bananas, rather than the unit price of an apple and the unit price of a banana and regarded a and b as prices of item. Hence, this study has revealed that the concept of a variable is more

sophisticated than teachers expect and it frequently becomes a barrier to a student's understanding of algebraic ideas. Consistent with Wagner's [17] observation we also noted here that the students experienced some difficulty in shifting from a superficial use of a to represent apples to a mnemonic use of a to stand for the number of apples.

The next question tested students' understanding quantitative comparison by means of the questionnaire item: *Which is larger $\frac{1}{n}$ or $\frac{1}{n+1}$, when n is a natural number? Justify your answer.* Incorrect quantitative comparison of two algebraic fractions occurred. 63% of the students substituted numbers in the algebraic expressions. They only compared the magnitudes of denominators, instead of comparing the whole fraction. They arrived at faulty conclusion that $\frac{1}{n+1} > \frac{1}{n}$. Students prefer to confirm relationships by numerical substitutions an observation similar to the observation made by Lee [18]. The students failed to realise that the reciprocal of a number is smaller than the number itself under certain conditions.

3.2.2. Students' errors and misconceptions on algebraic expressions

Some students completely lacked conceptual understanding of product of two variables. Leitzel [19] asserts the concept of a variable is more sophisticated and a barrier to understanding algebra. Students' struggles with the idea of a product of two variables were revealed by their responses to the item: *What does xy mean? Justify your answer.* Students' interpretation of the product xy show in Table 2.

Table 2. Students' interpretation of the product xy

Expected response	Examples of students' responses
x multiplied by y or x times y or $x \times y$	Expression
	Unknown
	Coefficients
	x by y
	Variables

Some students showed a limited conceptual understanding of the product of two variables and wrote among the incorrect responses expression, unknown and coefficients. Other students regarded the product of x and y are "variables". To obtain a revealing picture about students' thoughts the students who had provided responses such as expressions, were interviewed for further clarification. An excerpt of the exchange between first author's and one of the students called Blessing is now presented.

Researcher: What do you mean when you write x by y ?

Blessing: I divide x by y .

Blessing's utterance point to a serious misconception in the student's understanding of a product of two variables that was conceived as a quotient. Further exploration of the same idea was done by considering students' responses to the questionnaire item: *What answer do we obtain when we multiply $x + 3$ by 2.* Some students' are now presented in Table 3.

Table 3. Students' misconceptions in algebra

Expected response	Examples of students' responses
$2x + 6$	$x + 6$
	$x + (3 \times 2)$
	$2x + 3$
	$2x + 6 = 8x$ or 8 $2x + 3 = 5$ or $5x$ $x + 6 = 7x$
	$7y$
	$(x + 3)2$

Table 3 shows that in most cases students violated the distributive property, $x(y + z) = xy + yz$. For instance, the written response $x + 6$ indicates that the students multiplied 2 by 3 and x was not multiplied by 3. The violation of the distributive property was also seen in written responses such as $2x + 3$ where only the term x was multiplied by x and yet all terms inside the bracket should be multiplied by 2 for the distributive property to hold. Further, the written response, $2x + 6 = 8x$ reveals that unlike terms inside the brackets were added. Similar observations were noted when the students responded to the item: *Add $4x$ to 3.* Students also added unlike terms to obtain the incorrect answer $4x + 3 = 7x$. Hence, many

forms of invalid application of distribution law and adding of unlike terms were noted. In other cases procedural errors manifested when invalid equations featured when students formed equations unnecessarily instead of simplifying expressions. For example, the students forged the equation, $4x + 3 = 0, x = -\frac{3}{4}$. The questionnaire item: *Subtract $2x$ from 7* tested students understanding of algebraic expressions in word problems. Incorrect word matching led to reversal error. When the subtrahend was an algebraic term and minuend was a number in the word sentence, some students carried out the operation in a reversed order. Table 4 presented next gives a summary of common errors observed when students simplified tasks assigned.

Table 4. Errors made by students on tasks involving algebraic expressions

Written task	Expected response	Examples of students' responses	Description of errors observed
Add 7 to $3x$	$7 + 3x$	$10x$ $x = \frac{7}{3}$ or $-\frac{7}{3}$	Adding unlike terms Forming invalid equation
Simplify $p - 2c + p - 5p$	$-3p - 2c$	$-5pc$ or $6pc$ $7p - 2c,$	Simplifying unlike terms Failing to collect positive and negative terms
Simplify $x\left(\frac{a}{b}\right)$		$\sqrt[b]{x^a}$ or $x^{a-b}, \frac{ax}{bx}$	Multiplication of algebraic expressions confused with indices
Simplify $\frac{ax+bx}{x+dx}$	$\frac{a+b}{1+d}$	$\frac{a+b}{d}, a + \frac{b}{d}$ $\frac{x(a+b)}{x(1+d)}$ $x(a+b) \div x(1+d)$ $\frac{ax+bx}{x+dx} = \frac{a+b}{d}$	The number 1 obtained from $\frac{x}{x}$ treated as the integer 0 Cancelling not done. Simplifying Illegal cancellation

Table 4 shows the most prevalent errors among the students were adding unlike terms and formulating and subsequent solving of irrelevant equations. Forming of irrelevant equations confirms Wagner and Parker's (1984) equation-expression problem when students force expressions into equations and solve instead of simplifying. With regards to the error of adding unlike terms, it can be seen from Table 4 that the students failed to realise that an algebraic expression $7 + 3x$ can be a complete final answer cannot be simplified. This finding is consistent with Socas [20] who suggests that students sometimes engage with mathematical tasks without reflecting on meaning of embedded ideas in such tasks.

Incomplete simplification processes were a common feature of students' written responses when some students terminated the simplification of the algebraic expression somewhere in the middle of the process without realising the final answer. A possible explanation for the prevalence of such errors could be that the students had no adequate mathematical to deploy in order to allow them to proceed further. Limited mathematical resources were inferred from tendencies such as reproducing the problem again in a slightly modified format such as $7 + 3x = 7 + 3 \times x$ and then they terminated the procedure without completion.

Table 4 also reveals errors students made when they multiplied algebraic fractional expressions. For instance, for the task: *Simplify $x\left(\frac{a}{b}\right)$* , the major error observed was that the students multiplied both the numerator and the denominator of the fraction by the letter to get $\frac{ax}{bx}$. Sometimes they may assume that there is no denominator to the letter. It occurs when there is no visible denominator. They have difficulties in realising that a single letter can be represented by an algebraic fraction by taking the denominator as 1. Students assume that both numerator and denominator of the fraction should be multiplied by the letter. Errors emanated from previous learnt methods. Students misconstrued for a question involving exponents and then wrote incorrect answers as $\sqrt[b]{x^a}$ or x^{a-b} .

Finally, students' difficulties with algebraic expressions were manifested in the solution attempts to the task: *Simplify $\frac{ax+bx}{x+dx}$* . Common incorrect answers were $\frac{a+b}{d}$ or $a + \frac{b}{d}$ that emanated from processes in which in students correctly factorised out x in both numerator and denominator but failed to divide denominator and numerator by x leading to incomplete answers such as $\frac{x(a+b)}{x(1+d)}$ and $x(a+b) \div x(1+d)$. In other solution attempts, the students invented shortcuts when they just crossed out x s without going through the correct procedure of factorisation, a finding consistent with Young's [21] results.

3.2.3. Students' errors and misconceptions in functions

The task: *Given that $f(x) = x^2 - 1$, find $f(0)$, $f(2)$ and solve $f(0) = 0$* , was used to explore the kinds of misconceptions held students on functions. Table 5 presented next summarizes major misconceptions revealed by this study.

Table 5: Secondary school students' kinds of misconceptions on functions

Task	Expected solution	Incorrect response	Description of kinds of errors observed
Find $f(0)$ when $f(x) = x^2 - 1$	-1	$0^2 - 1$; 0 -1	Incomplete solution
Find $f(2)$ when $f(x) = x^2 - 1$	3	$(-2)^2 - 1$; 4 -1	Incomplete solution
Solve the equation $x^2 - 1 = 0$	1 or -1	$(x - 1)(x + 1) = 0$ $x = \pm\sqrt{1}$	Incomplete simplification
Find $f(0)$ when $f(x) = x^2 - 1$	-1	$f(0) = x^2 - 1 = 0$	Lack of knowledge of functional notation
Find $f(2)$ when $f(x) = x^2 - 1$	3	$f(-2) = x^2 - 1 = -2$	Limited knowledge of functional notation
Solve the equation $x^2 - 1 = 0$	1 or -1	$f(x^2 - 1) = 0$ $x^2 - 1 = 0$ then $\frac{0}{x^2 - 1}$	Conceptual error, student divided by $x^2 - 1$.

Table 5 reveals that students had a weak command of functional notation, could not simplify given expressions and made some conceptual errors. The misconceptions shown in Table 5 reveal that students experience difficulties with functional notations. These errors reflect students' fragile understanding of the notion of a function. These findings are in agreement with Nyikahadzoyi [22] who writes that the definition of a function seems problematic for some "A" level teachers and some students. Jones [23] posits that students have trouble with language of functions. Incomplete simplification of algebraic expressions also featured when students presented $(x - 1)(x + 1) = 0$ and $x = \pm\sqrt{1}$. Students failed to identify quadratic nature of this question. The idea of dividing both sides of an equation by some quantity manifested as a strong met-before as students wrote $x^2 - 1 = 0$ then incorrectly that led to $x = \frac{0}{x^2 - 1}$.

3.2.4. Students' errors and misconceptions in solving of equations

Student's solution attempts to the task: *Use the elimination to solve the simultaneous equations $+b = 5$; $a - b = 7$* . The students' solution attempts revealed that procedural errors occurred when students were in the process eliminating the unknown a from the two linear equations. The students added the two equations to eliminate a instead of subtracting or subtracting the two equations. This misconception emanated from their fragile understanding of simplifying integers and manipulating signs. They failed to realise they could still obtain the same solutions by adding or subtracting two equations. The topic, Algebra at secondary school level in Zimbabwe include the solving of linear and quadratic equations and so the students were asked: *Solve (a) $15 - 3x = 6$, (b) $x^2 - 4 = 0$, and (c) $x^2 - 4x = -4$* .

Prominent among the students were procedural errors that manifested in such forms as incomplete solutions, violation of addition and multiplication axioms in \mathbb{R} , inability to handle equations involving brackets, and errors and misconceptions connected to failure to apply the quadratic formula. For example, when solving the quadratic equation $x^2 - 4 = 0$ students had incomplete solutions and misused addition and multiplication axioms of the real field. For instance, an awkward formulation such as $x(x + 4) = 0$ was noted in students' attempts to solve the equation $x^2 - 4 = 0$. Further, incorrect answers such as $\sqrt{x^2} = \sqrt{4x}$, then $x = \sqrt{4x}$ or $x = (4x)^{\frac{1}{2}}$ and $\frac{x^2}{x} = \frac{4x}{x}$ then $x = 4$ were common. A possible explanation could be students' failure to negotiate the transition from one mental state to another which in turn may cause unstable behaviour when previous experience conflict with new ideas. For instance, attempts such $\frac{x^2}{x} = \frac{4x}{x}$ reveal that students failed to question the mathematical legitimacy of dividing by a variable.

Students' efforts to apply the quadratic formula to solve the equation $x^2 - 4x = -4$ showed that they failed to formula correctly. Some students wrote $= -b \pm \frac{\sqrt{x^2 - 4ac}}{2a}$. Students who resorted to use of the factorisation method wrote incomplete solutions like $(x - 2)(x - 2) = 0$ and, $= \frac{4 \pm \sqrt{0}}{2}$. In addition, students struggled to convert the quadratic equation

$x^2 - 4x = -4$ into standard form. Some students disregarded the use of essential brackets when substituting the negative value in the quadratic formula, $\frac{-4 \pm \sqrt{-4^2 - 4 \times 4 \times 1}}{2 \times 1}$. Nyaumwe [24] suggests that most the errors may be due to teachers' pedagogical practices. Methods used sometimes do not promote conceptual understanding

3.3. Research question 3 results

We restart research question three: *What are the possible sources of errors and misconceptions in the domain of school algebra?*

This study has revealed that errors and misconceptions in learning algebra are mainly content based and pedagogically driven. Content based sources of errors emanate from the abstract nature of algebra. This study showed that language difficulties or semantics of mathematical language can seriously cause misconceptions in learning algebra. For example, for a task with the narration: *for twice as much as x*, some students wrote x^2 instead of $2x$. Hence, instances such as the one just described reveal that students find algebra challenging because of the notations and language semantics use, particularly the functional notation was not well grasped as already discussed under research question 2. Algebra is a language of symbols and basic aspect of learning algebra is efficient use of symbols. The branch of Mathematics called algebra is very rich in symbols. The language of mathematics consists of symbols, terminology, notations, conversions, models and expressions that presented challenges to secondary school students involved in this study.

Pedagogically driven sources of errors and misconceptions in learning algebra pertain to the mode of lesson delivery in mathematics. The misconception emanating from the use of addition and multiplication axioms of \mathbb{R} as a field can be used to account for errors related to transposing of symbols and performing same operation on both sides as equivalent when solving equations. The recurrence of the error blinked to the mode of delivery where teachers do not insist of strategic use brackets (e.g., $\frac{-4 \pm \sqrt{-4^2 - 4 \times 4 \times 1}}{2 \times 1}$) when substituting negative values in the quadratic formula may be indicative of the teacher as the source of errors in the mathematics classroom.

3.4. Discussion of results

This study has revealed the importance of algebra and has reported on students' understanding algebraic ideas. Algebra is a powerful means of studying various mathematical structures in abstract form before applying the results to particular situations when they arise. Educators need to help students understand abstract concepts in algebra and look for relevant ways to introduce these concepts to students. Gillian [2] concurs that student best appreciate the abstract theory when they have a firm grasp of what is being abstracted.

Regarding students' conception of algebra the research revealed that errors are common in students' written work and that secondary school students have limited conception of algebra. Brodie [25] writes that errors and misconceptions are a result of mathematical thinking on the part of students; hence they are reasonable for students. The student's errors are actually natural steps to understanding. Students have limited understanding of algebra, in particular the notion of a variable was not well grasped. Students found algebra difficult and abstract to comprehend. The prevalent categories of errors were conceptual errors in variables and procedural errors in expressions, functions, equations and inequalities.

Concerning the kinds of errors and misconceptions, this study revealed that errors are either computational, algorithmical, procedural or conceptual in nature Young [21]. Conceptual and procedural errors were the most prevalent based on findings of this research. Consistent with Lockhead [26], errors and misconceptions are deeply rooted in the minds of students and are difficult to dislodge. Students' learning difficulties in algebra are attributable to concept learning. Conceptual knowledge is key to learning algebra. This study confirms findings by Van Lehn [27] who concur that gaps in conceptual knowledge lead to students using buggy procedures in solving problems in algebra. Student self-generated procedures or rules are sometimes faulty. These faulty rules have sensible origins that lead to faulty thinking in algebra that manifested as misconceptions in students' written attempts. Results from this study concur with Lins [28] who argue that tradition of arithmetic then algebra cause immense students' difficulties in learning algebra.

Regarding sources of errors and misconceptions, this study revealed that students can commit errors due to a myriad of reasons ranging from a data entry to calculation errors. The errors were due to carelessness, not understanding at all, confusing different concepts or interference from previous experiences. Misconceptions and errors in algebra emanated from the abstract nature of algebra; algebra content is symbolic, failing transition from object-oriented thinking to process-oriented thinking and failing transition from arithmetic to algebra. Consistent with Tall [29] this study revealed transition from one mental schemata to another may cause unstable behaviour when previous experience conflict with new ideas. This is also

consistent with Luneta [30] whose study led to the conclusion that errors and misconceptions are related but are different. Furthermore, the current study has revealed that errors emanate from misconceptions the student holds.

The main sources of errors and misconceptions were content and pedagogically related. The abstract nature of algebra and methodology of lesson delivery cause errors and misconception in learning algebra. Mathematics requires learners to think in terms of symbolic representation or abstract conceptualisation. Pedagogically, the teacher facilitates discovery of mathematics principles, patterns and relationships by students through inductive discovery and deductive discovery teaching approaches to mathematics teaching and learning.

Identifying what students may learn in algebra is of paramount importance. Effective algebraic thinking sometimes involves reversibility. It is the ability to undo mathematical processes as well as do them. It is the capacity not only to use a process to get to a solution, but also to understand a process well enough to work backward from the answer to the starting point. Students should have the capacity for abstracting from computations. This is the ability to think about computations independently of particular numbers used, thus generalising arithmetic. One key characteristics of algebra has always been abstractness. Abstracting system regularities from computation is when thinking algebraically involves being able to think about computations freed from the particular numbers in arithmetic.

4. CONCLUSION

A fundamental principle underlying constructivist approach to learning mathematics is that a student's activity and responses are always rational and meaningful to themselves, no matter how bizarre or weird they may seem to others. One of the teacher's responsibilities is to determine or interpret the student's rationality and meaning. Before constructivism, teachers often had negative feelings about the errors the students would make regarding them as unfortunate events that need to be eliminated and possibly avoided at all times. Students make errors in the process of constructing their mathematical knowledge. However, regarding errors as valuable sources of students' thinking replaced the strategy of more drill and practice. As teachers, it is difficult to escape from students' errors, so it is worthwhile finding out why students make errors in the first place and often continue to repeat the same errors. Errors become entrenched in students' cognitive structures, so error analysis is the first step towards doing something relevant to remove the cause of the errors.

Consequently, teachers are encouraged to embrace the errors and engage with them rather than avoid them. Teachers need to respond to students' errors in ways that involve understanding of students' thinking behind the error. Teachers should shift their minds from understanding of students' errors as obstacles to learning mathematics to understanding errors and misconceptions as an integral to learning and teaching mathematics. Understanding errors is a vital part of correcting them. Mathematics educators should shift from the tradition of arithmetic then algebra. An early introduction to algebraic reasoning is strongly recommended. Students should develop rational thinking with number senses to assist with transition to literal symbols as suggested by Stephens [31].

The ideas from constructivism should inform packaging of content when employing inductive and deductive teaching approaches. It is the role of the mathematics educators and policy makers to link students' relevant prior knowledge and experiences with existing knowledge when developing new knowledge of mathematics. Presentation of content should be done in a manner that does not cause abrupt or drastic shifts in their cognitive models. We argue that new knowledge should allow students to operate within zone of proximal development. Hence, students should be weaned from practice of receiving knowledge from the teacher. The students should discover patterns, extrapolate from materials presented by them while the teacher guides the students in line with Prince [7]. On the basis of findings from this study, the researcher strongly recommends in-service of teachers. Teachers are encouraged to organise school-based or cluster-based workshops to cross-pollinate mathematics pedagogical content and demystify math phobia and algebra mythology.

Exposing students to solving real life mathematical tasks is essential in developing problem solving skills. According to Polya [8] there are four phases through which a problem-solver proceeds in order to solve a confronting problem successfully. These are sequentially, understanding the problem, devising a plan or deciding on an approach for tackling a problem, executing a plan, looking back at the problem, the answer and how it was obtained. This fascinating area deserves further exploration. Systematic errors are recurrent wrong responses methodically constructed and produced across time and space. They were uncovered. The test was not repeated. Research studies into systematic errors in algebra can go a long way in improving the teaching and learning of secondary school mathematics.

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