What should be the object of research with respect to the notion of mathematical proof?

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ABSTRACT

Despite its central place in the mathematics curriculum the notion of mathematical proof has failed to permeate the curriculum at all scholastic levels. While the concept of mathematical proof can serve as a vehicle for inculcating mathematical thinking, studies have revealed that students experience serious difficulties with proving that include (a) not knowing how to begin the proving process, (b) the proclivity to use empirical verifications for tasks that call for axiomatic methods of proving, and (c) resorting to rote memorization of uncoordinated fragments of proof facts. While several studies have been conducted with the aim of addressing students' fragile grasp of mathematical proof the majority of such studies have been based on activities that involve students reflecting and expressing their level of convincement in arguments supplied by the researchers, thereby compromising the voice of the informants. Further, research focus has been on the front instead of the back of mathematics. Hence, there is a dearth in research studies into students' thinking processes around mathematical proof that are grounded in students' own proof attempts. Therefore current investigations should aim at identifying critical elements of students' knowledge of the notion of proof that are informed by students' actual individual proof construction attempts.

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1. INTRODUCTION

Mathematical proofs play an important role in generating mathematical knowledge, promoting, and fostering thinking among students as asserted by Duruk [1]. While the concept of mathematical proof is a vehicle for mathematical thinking as has been noted by Duruk [1] other researchers in mathematics education such as Varghese [2] have reported that many students face overwhelming challenges with the notion of mathematical proof to an extent where the concept of mathematical proof has failed to permeate curricula at all scholastic levels. Mukuka [3] found that serious challenges students face with proof and proving are perhaps attributable to the kinds of views on proof held by students as well as their lack of confidence to engage in proof arguments.

Mathematical proof is a powerful tool for generating important insights into why a mathematical proposition is true or false. In other words, mathematical proving promotes mathematical thinking which, according to Jonassen [4], is a form of formulating and weighting the argument for or against a course of action, a point of view, or a solution to a problem. However, despite its central place in the curriculum, researchers such as Selden [5] have found that student teachers have serious difficulties with proof and proving. Studies have revealed that many undergraduate student teachers lack adequate understanding of the notion of mathematical proof or have inadequate conceptual knowledge of how to construct proofs Ko [6]. So those teachers are less likely to develop the subject in a persuasive manner when they teach their future students because the competence of a teacher matters when it comes to enabling students to acquire reasoning and proof writing skills Mukuka [3].

Following Harel [7] a mathematical proof can be described as a particular argument one produces to ascertain for oneself and convince others about the truth-value of a mathematical proposition. Proving is the truth seeking exercise which is defined as the mental act that a person or a community employs to get rid of their doubts about the truth of a mathematical conjecture. In this sense Goethe [8] suggests that proving is not confined to the removal of one's doubts through axiomatic means but encompasses students' enculturation of students into socio-mathematical norms of how correctness is determined in mathematics___ an idea also suggested by CadawalladerOlsker [9]. The socio-mathematical norms with regard to mathematical proof construction include Goethe's [9] notion of analytic approach to proving. An analytic approach is when an individual proving a mathematical statement traverses the inductive-deductive continuum in a versatile and strategic manner. By traversing the inductive-deductive continuum that way reference is made to cases where a prover constructs an axiomatic argument to validate a true mathematical proposition and or alternatively produces a counter example to refute a false mathematical statement.

Students' efforts to justify mathematical statements yield exploratory arguments that have certain characteristics. The characteristics possessed by arguments produced by provers in their efforts to validate or alternatively refute a mathematical claim give rise to the notion of a proof scheme. The mental act of eliminating one's doubts about the truth of a mathematical conjecture leads to arguments that have persistent characteristics. For example, an individual can show a proclivity to use specific examples in establishing the accuracy of a mathematical assertion. According to Harel [7] persistent characteristics of proofs produced by a student for the purpose of ascertaining for her/himself or persuade others about the truth or falsity of a mathematical assertion constitute what is called a proof scheme. Hence, in line with Oflaz [10], we describe a proof scheme as a collection of persistent cognitive characteristics of the proofs one produces. What kinds of proof schemes should be held by students in order to allow the notion of proof to permeate mathematics education curriculum?

Efforts to establish students' schemes of argumentation around the idea of a mathematical proof involve unraveling kinds of students' thinking. Therefore investigating students thinking during proving becomes crucial because proofs are at the heart of mathematics as they promote critical thinking. What then should be the object of research in order to illuminate the kinds of students' thought processes during proving? In other words, how can we develop an understanding of the ways in which students think of the concept of mathematical proof. Prior to addressing this question we examine some of the existing literature ideas on mathematical proof and proving.

2. CRITICAL REVIEW

This section discusses some literature examined in connection with students' learning of mathematical proof. Stylianou [11] and Varghese [2] concur that studies on students' understandings of mathematical proof run the gamut from university level where studies have involved pre-service mathematics education student teachers and mathematics majors to the secondary level. The thrust at secondary school has been on proof validations where students largely reflect and evaluate arguments supplied by the researcher as can be exemplified by Stylianides [12] and by Martin [13] on pre-service elementary teachers. Some studies have uncovered a variety of phenomena regarding the ways students comprehend and appreciate the notion of mathematical proof that include holding an empirical conception of mathematical proof Harel [14] and Stylianides [15]. Other studies, such one by Stylianou [11], clarify the kinds of patterns that exist between students' schemes of argumentation and their problem solving strategies. Previous studies have also revealed student teachers' superficial understanding of the notion of a counter example. In the following section, we review some of those studies and in the process try to justify why it is critical for the student's voice to be a focal idea of current research efforts on mathematical proof and proving.

First we look at an article by Weber [16]. With the goal of establishing research mathematicians' motivations and strategies for reading proofs, Weber and Mejia-Ramos used semi-structured interview guides to capture reasons research mathematicians read proofs and how they behave when they try to comprehend published proofs. Weber [16] applied content analysis technique to verbatim transcriptions of semi-structured interview data thereby uncovering many reasons for research mathematicians' motivations and strategies for reading published proofs. The motivations include searching for new ideas and checking for utility and originality of ideas generated in theorems read.

While we acknowledge that these are important insights with respect to efforts intended to understand the learning of mathematical proof we would like to take cognisance that these findings were based on participants' evaluations or validations of proofs supplied to them and not on their own productions. Azrou [17] has, however, emphasized that it is important to analyse students written tasks produced individually. Hence, we think, too, that the object of current research should be on exploring students' thinking about mathematical proof based on students' own voices, that is, students' own proof attempts.

Second, we consider a report of a research study by Stavrou [18] on *common errors and misconceptions in undergraduate mathematical proving by education undergraduates.* The study took place in a context of two proof laden courses namely, Number Theory and Abstract Algebra, and had two main goals. One of the goals was to identify common errors and misconceptions made by the students when proving. Another major goal was to determine how students' proof behaviour would change when student teachers were made aware of those errors. While Stavrou [18] can be hailed for making efforts to involve students in proof constructions during data collection, we somehow question the data collection technique used. During data collection respondents were allowed to work on assigned proof tasks as homework. Working at home is likely to have compromised independent reasoning by student teachers since it is most likely that students could present "workings" from other sources as data. Further, novelty of proof tasks assigned was also likely to have been compromised because Stavrou [18] stated that participants proved routine statements covering basic Number Theory and Abstract Algebra. There is therefore need to explore students' mental constructs around the notion of mathematical proof on basis of students' independent reasoning.

Third, we comment on Duruk's [1] research report on *prospective mathematics teachers' difficulties in doing proofs and causes of their struggle with proofs*. Some of the difficulties students have with proofs covered in Duruk [1] and Kaplan's literature review section are:

a. Not knowing how to make a proof structure using definitions
b. Being unable to use concept images
c. Not knowing how to begin the proof construction process.

Duruk [1]'s main goal was to uncover prospective teachers' difficulties with mathematical proof as well as revealing reasons for such difficulties from the perspective of actors, i.e., the prospective teachers. A case study was considered strategic for the purpose of studying prospective teachers' difficulties and their understanding of causes of such difficulties.

To gather data Duruk [1] asked prospective mathematics teachers to prove a theorem drawn from topology real numbers. Precisely the students were asked to prove that "every neighbourhood is an open set." Students' responses were in written form. Similar to Stavrou's [18] study described earlier Duruk [1] instructed respondents to present their proof attempts in written form. So, Duruk [1] can also be hailed for making student teachers respond in written form because Manilla [19] suggest that documenting students' proof attempts have been suggested to be effective in illuminating their thinking. However, we question as well novelty of the task, given that it appears to be routine since it is one of the basic ideas in most mathematical literature texts on topology of the real line.

Duruk's [1] study revealed that student teachers encountered extreme difficulties with proof to such an extent that they were unsuccessful in proving. One of the difficulties encountered in proving related to use of definitions. The prospective teachers failed to state definitions correctly and failed to organize the definitions into a valid proof. The prospective teachers also failed to use mathematical language correctly. For example, students confused notations and language because they did not understand the theorems or propositions. In that situation pre-service teachers failed to pay attention to the scope of the statement to be proved. The consequences of such superficial understanding of proof concepts resulted in pre-service teachers resorting to proving other propositions instead of the statement in question.

Fourth, Ugˇurel's [20] study on *pe-service secondary teachers' behaviors in the proving process is examined.* The major goal of the study was to generate insights about the kinds of proof behaviours revealed by pre-service teachers when they proved a given proposition. Ugˇurel [20] studied 15 volunteer pre-service teachers of whom 5 were male and 10 were female. To solicit data each pre-service teacher was asked to prove a given proposition on the chalkboard by thinking aloud as the student teacher presented the proof.

However, with Ugˇurel [20] data collection was a one day event which made it impossible for the study to derive benefits associated with prolonged engagement in the research setting that would have allowed Ugˇurel [20] to uncover less visible aspects about students' proof behaviour difficult to unravel in a single day as asserted by Maxwell [21]. Further, collecting data on a single day is likely to cause conversational fatigue on the part of participants and even the researchers themselves.

While we appreciate efforts by Ugˇurel [21] to apply the emic approach, that is, to determine students' proof behaviour by engaging students in think aloud interview protocols, the fact that data collection took place in a single day and that just a single task was used in data collection point to the need for further studies that allow the voice of the student to be heard in order to illuminate the kinds of student's schemes of argumentation during proving. Hence, more studies in which the students' voice is prominent are vital to avoid what Hennink [22] call a mere passing mention of an event with respect to proof and proving in mathematics education.

Fifth, another piece of literature reviewed is a study that focused on *proof validations of prospective secondary mathematics teachers* by Bleiler [23]. The study had two aims, one of which was to evaluate the effectiveness of an instructional sequence in improving pre-service teachers' abilities to validate arguments produced by high school student teachers. Another aim of the study was to determine the sort of errors pre-service mathematics teachers attend to when validating mathematical arguments purported to be proofs by high school students.

Bleiler [23] then designed and implemented a sequence of intervention activities with the intent to increase pre-service teachers' awareness and skills in validating mathematical arguments. The design of the sequence of activities was informed by two critical ideas drawn from these researchers' survey of literature on students' behaviour during proof validations. First, students showed a tendency to use inductive arguments to prove mathematical propositions. Second, the instructional activities were influenced by the finding that teachers tend to focus on local components of an argument rather than focusing on reasoning and the logic sustaining the entire argument. The consequences of focusing on the specifics as opposed to considering the proof as a holistic entity include the following proof behavioural tendencies. In some cases a bi-conditional mathematical statement was considered valid when a proof of the implication statement $p \Rightarrow q$ has been provided without proof of its converse, $q \Rightarrow p$. The purpose of the study by Bleiler [23] was thus to evaluate the effectiveness of an instructional sequence that gave particular attention to these limitations in students' proof behaviour. Precisely, the study sought to determine whether the instructional intervention could ameliorate the proclivity by students to use particular instantiations and the tendency to focus on local aspects instead of considering the proof as a holistic entity.

Bleiler [23] reported that the instructional sequence increased students' awareness of the fundamental limitation of the empirical proof scheme. The study also revealed that students did not draw meaning from mathematical objects constructed. For example, some students did not provide a justification why the fraction $(ky+x)/y$ is irrational when given that x is irrational and y is a rational. Other errors also include use of imprecise or incorrect definitions and violation of the proof framework when proving. While Bleiler's [23] findings are critical ideas that can provide a useful window through which students' thinking of mathematical proof and how such thinking evolves can be determined, it can be emphasised that the study by Bleiler [23] used data based on pre-service teachers' proof validations of arguments supplied by the researchers. This justifies the call for more studies based on students' own proof constructions instead of reflecting on arguments supplied by the researchers.

Sixth, Recio [24] with the aim of determining the nature of arguments student teachers find convincing in different learning contexts such as daily life, experimental sciences, and logic and mathematical foundations studied year one mathematics students at University of C'ordoba in Spain. The main conclusion pointed to students' difficulty with axiomatic proofs. This finding can be explained by the fact that the study was done at transitory phase, that is, a few days after commencement of university mathematics studies when the students had just passed a stage where mathematics learning was characterized by low intensity of mathematical proof activities.

Finally, Varghese [2] took a case study approach involving 17 prospective mathematics student teachers to examine both students' conceptions and their ability to construct proofs of given mathematical statements. All the students were mathematics majors who had completed undergraduate courses and just commenced studies on pre-service teacher education. The study findings regarding students' conceptions of mathematical proof were that (a) the dominant meaning of proof was one in which proof is viewed as serving the verification purpose (9 out of 17), and (b) the least number of responses was in the category where proof was considered by students as a tool for explaining and discovering of mathematical knowledge. Regarding the proof construction process, 13 out of 17 suggested a teacher guided step-by-step procedure as the way to complete proof tasks. Varghese's study revealed the dominance of the external conviction proof scheme, specifically the authoritarian proof-sub-proof scheme where the teacher and the textbooks are authorities for the right answer. This is a lower cognitive scheme of argumentation with regard to proof construction. What should then be the object of research in undergraduate mathematics education in order to allow students to progress to higher cognitive proof scheme categories such as the axiomatic proving?

The call for studies that focus specifically on students' abilities to construct mathematical proofs has been made by many researchers after realizing the dearth in studies with an emphasis on mathematics education undergraduate students' proof construction competences. That there is scarcity of studies based on students' own proof attempts has been emphasized by Ozdemir [25]. It can be argued that a potential factor for students' struggle with mathematical proof is that the students lack a deep understanding of mathematical proof in order to be able to solve problems involving the concept of mathematical proof. Yet, effective teaching and learning of any topic in mathematics cannot take place without engaging in some sort of proving because proof justifies why a mathematical idea is valid or can be refuted.

Students' difficulties with mathematical proving point to the need for detailed profiling and careful assessment of their proving abilities. How then can detailed profiles of students' proving competences be developed? In other words what should be the focus of researchers' current efforts in order to develop a more revealing picture about the manner in which prospective teachers think around the notion of mathematical proof? Let us now try to address these questions in the next section.

3. DISCUSSION

As a tool for mathematical learning proof leads to mathematical understanding as the prover explains a theorem and the content it concerns. So the notion of mathematical proof is central to mathematical learning. Despite the central role mathematical proof plays in teaching and learning of mathematics at all scholastic levels, students have exhibited a fragile understanding of the concept. For example, Cusi [26] have revealed that mathematics teachers who would have completed several undergraduate and postgraduate studies still struggle to produce proofs autonomously.

Furthermore, with many studies emphasis has been on checking a given proof is indeed true. However, the essential activity regarding the learning of mathematics should be constructing or finding a mathematical proof. In other words, there is a distinction between checking a given argument for its accuracy and doing proofs. There is therefore need for clarity on this distinction. Doing proofs is described as a cognitive act performed to eliminate doubts of an individual or community regarding the accuracy of a mathematical claim Iskenderoglu [27]. Thus, doing a proof refers to those efforts intended to eliminate students' doubts and should be based on these students' own voices. However, in stark contrast, Cirillo [28] have reported that proof-oriented instructors rarely ask students to compose proofs of statements or tasks students have not seen before so that students do not engage in independent reasoning. Weber [16] describe reading a mathematical proof as the act of examining an existing argument to check its validity or for the purpose of understanding the essence of the given argument. Therefore the object of research on mathematical proof should be to develop an understanding of students' mathematical thinking as they construct proofs as opposed to reading proofs. When proof is conceived in terms of s student's own construction attempts as opposed to reading an existing argument a mathematical proof has the potential to play an important role in developing and shaping the student's thinking.

There is overwhelming evidence of students' difficulties with producing proofs. One way of overcoming these difficulties would be by investigating the cognitive processes of students during proof construction through research activities that emphasize the back as opposed to the front of mathematics. Further, discussions and research efforts on mathematical proof and proving have shown an orientation towards the front of mathematics. Metaphorically, the front of mathematical proof, refers to the conception of mathematical proof as presented in journals, textbooks, lecture notes while the back of mathematics is used to refer activities that take place in the workshop of a research mathematician where there is an interplay between both syntactic and semantic approaches to proof making, which is similar to the analytic mode of proof production proposed by Goethe [8]. However, not much has been explored with respect to what constitutes the back of mathematics. Hence, in addition to fostering proof construction efforts among students, researchers should also pay attention to research activities with a bent towards the back of mathematics. Activities of the back of mathematics should receive emphasis in current research efforts to allow proof to permeate the mathematics curriculum in order to allow students to develop an analytic conception of mathematical proof.

Commenting on the typology of warrant types involved in proving, Boero [29] noted that while empirical justifications and structural-intuitive arguments are useful in some stages of conjecturing and proving they do not appear in the products of these two processes, that is, conjectures and proofs of theorems. The point drawn from this piece of literature in connection with students' superficial understanding of mathematical proof is that exposing students to the front of mathematics, which is typical of undergraduate teaching and learning of proof, obstructs conjecturing and proving activity that is essential in revealing students' thinking processes. An elaboration of analytic proofs is considered next for purposes of elucidating what the focus of research should be with respect to proof and proving.

In the workshop of a mathematician who writes proofs analytic proofs are a prevalent feature Goethe [8]. An analytic prover strives to reach mathematical conviction by using a mixture of both deductive and induction moves. In a deductive move, the prover proceeds from axioms and then utilizes logical rules to lead to a conclusion. Induction is construed in this context to refer to instantiations of mathematical ideas such as graphs, tables, figures and other structural-intuitive displays of ideas pertinent to the mathematical proof task or claim. The analytic approach allows the prover to simultaneously employ axiomatic and induction means to build an argument for or against a mathematical proposition. The interplay between intuition (induction) and formal (axiomatic) allows students to develop a comprehensive perspective of mathematical proof. In other words, developing an analytic conception of mathematical proof allows student teachers to test and compose proofs.

Furthermore, Maya [30] have noted that students can follow a proof when explained by their instructors in class but would not be able to compose proofs themselves. There is a paucity of research into the typology of warrant types that are typical of the analytic understanding of mathematical proof at undergraduate level. Little is known particularly in our local Zimbabwean context about how students conceptualise mathematical proof based on their actual voices. Therefore, the object of current research should include an attempt to investigate student teachers' reasoning based on their actual proving efforts. We reiterate that research activities should reflect an orientation towards the back of mathematics in order to develop a more revealing picture of students' thinking about mathematical proof.

Kidron [31] have argued on the basis of extensive research carried out by many mathematics educators on matters related to proving that proofs are at the heart of mathematics. However, as earlier noted those research studies have tended to focus on students' ability to reflect and validate proofs supplied. Once again we reiterate that there has been a scarcity of research that addresses how students go about constructing proofs. This dearth in research based on students' actual proof construction efforts has seen many researchers advocating for more in-depth studies into students' conceptions of mathematical proof that are based on students' personal constructions. For instance, Marrioti [32] has pointed out that further investigation is needed into students' own active individual autonomous proof constructions with particular emphasis on analysis of the cognitive processes involved in proving. It can be inferred from Mariotti's [32] assertion that more studies in which the voice of the student is prominent are critical.

Further, Selden [33] even suggest that more studies grounded in students' actual voices could be also done in examining the processes involved in proof construction. Even in circumstances in which proof-oriented instructors try to involve students in proof production, Selden [33] have noted that not much time is devoted to helping students to learn how to construct proofs. Rather, emphasis is on producing fragments of proofs or original proofs presented as lecture notes in a neat fashion with little or no resemblance at all to the back of mathematics where an analytic approach is employed in producing these proofs. Hence, while it has been useful to generate knowledge about students' conceptions of mathematical proof through proof validations it is also thus crucial to gain insights about undergraduate student teachers conceptualisations of mathematical proof from their actions and behaviour as they engage in proof constructions.

As noted earlier, proof is essential for deep mathematics learning. We argue that mathematics students' understanding and ability to construct proofs are not only important for their own learning but it is also crucial for the future high school teachers to help learners learn how to construct proofs. Hence, in order for the student teachers to be able to promote proving abilities among their future students, they need to be able to build a strong foundation of the proof concept. Although it is now documented that constructing a mathematical proof is a complex process that calls for a large expanse of knowledge and skills and is determined by the learning contexts, many such studies have based their conclusions on arguments students find convincing (convincement issues) and validation of proofs supplied to the participants by researchers. This observation was also made by Imamoglu [34] who stated that over the past decades researchers have focused on proof validations which Selden [33] define as readings and reflections on proofs to check their correctness. These proof validations are carried out on proof texts supplied by researchers. We argue that investigations into students' understanding of mathematical proof should be grounded in students' own efforts. Apart from personal observations, available literature sources suggest the need for in-depth studies into students' conceptualisations of mathematical proof since there are a few empirical studies on how well students understand proofs.

We emphasize here that most studies have focused on students' abilities to comprehend given mathematical proofs, that is, proof validations, yet research on learning of mathematical proof and associated difficulties must be based on what students really do by themselves, rather than relying on students expressing their conviction levels on the validity of arguments supplied by researchers. Hence, the current research efforts should respond to the dearth in studies on ways in which individual students think around the notion of proof. To fill this gap studies should be designed to develop an understanding of undergraduate students' behaviour as they engage with mathematical proof tasks.

Furthermore, while mathematics students have shown a preference for deductive proofs-a higher level proof scheme-studies by Azrou [17] have revealed that these students were not able to compose axiomatic proofs by themselves. When those students were asked to produce proofs of tasks that required use of formal deductive means they resorted to particular instantiations. This sort of proof behaviour reveals a discrepancy between what students produced as proofs and what they chose as closest to their preferences in terms of convincing arguments.

It can be inferred that it is often easier to read a proof than to produce a proof. This provides further evidence about the limitation of relying on proof readings as basis for measuring students' competences at constructing proofs. Hence, we argue that an understanding of the mental processes involved in proving and how proof schemes develop among undergraduate student teachers merits close attention and one way of ascertaining students' proof competencies is by examining their proof productions.

One of the primary goals of mathematics instruction is for students to develop standards of proving and conceptions of mathematical proof that are held by research mathematicians. Hence, research on students' mathematical proof competences should involve measuring discrepancies between students and mathematicians' conception of justification and proving processes of mathematics statements. Therefore, the object of research with regard to mathematical proof should be in response to the call to bring students' proof experiences as close as possible to the practice of mathematicians. Thus, the intention of the focus of current studies should aim to determine students' thinking abilities on justification and proof by addressing the questions of how student teachers go about constructing proofs and how the mathematical object (proof scheme) evolves.

A careful analysis of critical elements of students' proof schemes and the individual's thinking and reasoning around the notion of mathematical proof is needed in current researchers' efforts to raise students mathematics proof competence levels. The object of research with regard to mathematical proof should therefore show an inclination towards efforts intended to transform the students' view of a mathematical proof as a special form of producing written work to a conception of proof as a vehicle for producing reliable explanations for the accuracy (or lack thereof) of mathematical propositions and hence a means of achieving understanding.

Mathematical proof is an essential tool in learning mathematics. Understanding how student teachers conceptualize mathematical proof is an essential consideration for thinking about how to teach proof. Developing an understanding of the nature of students' conceptions of mathematical proof will in turn inform the process of identifying appropriate learning opportunities for students to engage in during learning.

Stylianou [11] has recommended the necessity for more studies that illuminate processes students use when they engage in constructing proofs. In-depth studies that would uncover salient features of students' thinking around the notion of mathematical proof could produce a knowledge base for understanding undergraduate student teachers' conceptualisations of mathematical proof. Identifying critical elements of students' knowledge involved in proving can provide a clearer picture of teachers' knowledge of situations for proving. The term knowledge of situations for proving is identified as part of teachers' knowledge about proof for teaching involved in the mobilization of proving opportunities for students.

In Zimbabwe there is paucity in research on undergraduate students understanding of mathematical proof, particularly studies with grounding in students' individual proof attempts. In the teaching and learning of proof-laden courses such as Real Analysis at undergraduate level, the sequencing of instruction has generally followed the format "definition-theorem-proof." In addition, assessment modes have focused on how students' comprehension of a given mathematical proof can be measured through efforts such as reproducing deductive arguments from lecture notes or modifying the proof slightly to prove on analogous proof, e.g., lecture notes on the theorem: *The least upper bound of a subset of* $\mathbb{R}$ *that is bounded above is unique* can be modified slightly by the student to prove analogous statement: *The greatest lower bound of a subset of* $\mathbb{R}$ *that is bounded below is unique*. It can be noted that these types of assessment only serve to provide a superficial understanding of mathematical proof because, to accomplish the proof, the student proceeds in a secure ritual manner by just modifying slightly lecture notes on the uniqueness of a least upper bound of a subset of real numbers that is bounded above.

It can be seen from the foregoing discussion that asking students to express their level of conviction in arguments and/or proofs supplied by the researcher does not do enough to involve students in the manifold of activities and processes involved in proving. Furthermore, research studies have revealed that although the processes of validating and composing a mathematical proof are interwoven, it is more difficult to construct than to read a proof. So engaging students in proof constructions is more likely to generate more insights into the cognitive processes involved in proving. So, there is need for research based on students' own proving efforts.

To further emphasize the need to develop an understanding of student teachers' conceptualisations through their own voices, we draw ideas from Boero [28] and Weber [34]. Weber [35] articulated three activities involved in argumentation process that mathematicians engage in when proving: constructing a novel argument, presenting an already existing argument and reading an available argument. We reiterate that constructing a novel argument is more likely to generate more insights into students' thinking processes than any of the other two activities involved in mathematical proving and hence novel proof tasks should form an integral component of major focal activity of current research on mathematical proof. We emphasize the point made earlier that constructing a proof is different from reading an available argument.

Consequently, the current research push should be to build on existing research studies on students' understanding of mathematical proof by examining undergraduate mathematics education students' mental constructs around the notion of mathematical proof as well as developing an understanding of students' experiences within the universe of discourse (phenomenon of interest), which is mathematical proof on the basis of students' actual voices.

4. CONCLUSION

The foregoing discussion has shown that profiling and evaluating students' proving competences should be based on students' actual proof productions instead of on proof validations. This way research is likely to make significant contributions in understanding how students think and how they communicate their thinking. Developing such ideas is critical in efforts to improve the learning of mathematical proof. Determining student teachers' thinking of proof construction is crucial because mathematical proof is essential for inducing flexibility in learners' thinking. A deep understanding of mathematical proof by prospective mathematics teachers should be developed because a weak command of subject matter makes these teachers feel insecure in their instruction and consequently the concept of mathematical proof may not receive the emphasis desired by curriculum developers as was also suggested by Varghese [2].

A comprehensive conception of mathematical proof should involve understanding proving as a process of constructing an argument or an explanation that justifies the truth-value of a given mathematical statement. This definition of mathematical proof points to the prominence of the student's voice in proving mathematical conjectures as opposed to validating arguments supplied by researchers. Further, the student's voice should be heard in research activities that show an orientation towards the back as opposed to the front of mathematics. Therefore the object of research with regard to proof and proving should be based on students' proof attempts in order to contribute to an improvement in the learning of mathematical proof that would result in a shift in emphasis from regurgitation of routine instructor's notes to construction of meaning through students' engagement in processes of conjecturing and proving.

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