Journal of Education and Learning (EduLearn)
Vol. 20, No. 4, November 2026, pp. 2102~2115
ISSN: 2089-9823, DOI: 10.11591/edulearn.v20i4.27206

Semiosis formal construction of limit concepts: a study on students' comprehensive triadic understanding

Beni Asyhar[1,2], Purwanto[1], Rustanto Rahardi[1], Sudirman[1]
[1]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Negeri Malang, Malang, Indonesia
[2]Department of Mathematics Education, Universitas Islam Negeri Sayyid Ali Rahmatullah Tulungagung, Tulungagung, Indonesia

Article Info

Article history:

Received Dec 28, 2025
Revised Mar 4, 2026
Accepted Jun 6, 2026

Keywords:

Construction
Formal definition of limit
Peirce's triadic
Semiosis
Semiotics

ABSTRACT

This study applied Peirce's triadic semiotic framework to map students' representational transitions in constructing the formal definition of limit, an approach that has rarely been employed in limit research. Meanwhile, the prior studies documented students' difficulties and instructional interventions, few explicate the internal semiotic mechanisms underlying the shift from visual intuition to the epsilon-delta formulation. Using a qualitative case study approach, this study examined students' semiotic processes in formally constructing the concept of limit through written tests and interviews. The test was administered to 21 students for 120 minutes to assess their understanding of the limit concept, meanwhile interview with 2 students was conducted for 90 minutes for each student to explore their semiotic processes. The results indicated that the students constructed the limit concept in a formal manner through the sequential stages of representamen, object, and interpretant operating in a triadic and comprehensive. Multimodal representamens facilitate the transition toward the formal epsilon-delta conception; the object was construed as the function's behavior near a point; and the interpretant emerges as a coherent formal understanding. Findings show that multimodal representamens facilitate students' transition from intuition to rigorous epsilon-delta reasoning.

Corresponding Author:

Rustanto Rahardi
Department of Mathematics Education, Universitas Negeri Malang
St. Semarang 5 Malang, East Java, Indonesia
Email: rustanto.rahardi.fmipa@um.ac.id

1. INTRODUCTION

The concept of limits is fundamental to the study of calculus, providing the formal foundation for derivatives, integrals, and continuity through the idea of approximation. The derivative is defined as the limit of the ratio of differences as the interval of change approaches zero, while the integral is understood as the limit of the Riemann sum as the partition becomes finer [1].

A comprehensive and integrated understanding of the concept of limits is essential for first-year calculus students. This understanding encompasses intuitive understanding, visual representation skills, mastery of symbolic notation, and skills in constructing and evaluating formal arguments based on the epsilon-delta definition [2], [3]. Through this approach, students are expected not only to memorize symbols, but also to be able to translate the idea of "approximation" that often appears in graphical contexts or everyday descriptions into precise quantitative statements. This involves understanding the use of the universal (∀) and existential (∃) quantifiers in the context of limits, which are fundamental in supporting the learning of calculus and mathematical analysis [2].

The epsilon-delta definition provides the formal logical structure necessary for defining limits and constructing rigorous mathematical proofs [2], [4]. Mastering the definition of epsilon-delta not only trains symbolic manipulation but also strengthens understanding of the fundamental principles of calculus and its implementations in various fields. This approach enriches mathematics education by connecting theory and practice to prepare the students for their future academic and professional challenges [5]–[7]. Furthermore, the ability to transfer between visual, verbal, and symbolic representations is a key element in enriching mathematical meaning [8]–[11] and supporting a deep understanding of limits [2], [12]. The use of various external representations of mathematical concepts, such as graphs, tables, and algebraic notation, can significantly influence how students think about the concepts [13], [14]. This suggests that when students are asked to translate between different representations, they not only practice technical skills but also deepen their conceptual understanding of limits.

However, despite its central role in calculus, students' struggle with the formal epsilon-delta definition, as intuitive and formal reasoning often fail to integrate. The results of those previous studies showed a sharp contrast with the expected conditions. The epsilon-delta approach in calculus often presents a challenge for students in converting intuitive understanding into formal form. The difficulties faced by students are largely due to the lack of experience in translating conceptual concepts into formal notation and arguments required at advanced levels of mathematics [15]. Curricula that tend to emphasize technical aspects rather than building conceptual understanding exacerbate this problem, so that students have difficulty in connecting intuition with formal symbolism and experience confusion when they are faced with proofs [15], [16]. Consequently, the inability to formulate formal definitions impacts the understanding of mathematical analytical structures and their implementations in other fields [17]. To bridge this gap and improve students' readiness in making the transition from intuitive to formal understanding, it is recommended to use a practice-based learning approach, problem-solving, and the use of visual aids and educational technology [18].

Previous studies consistently report a recurring pattern of difficulty, with students tending to rely on visual interpretation and experiencing logical barriers when entering the realm of formal proof [15]–[17]. Although various instructional interventions [18] including dynamic visualization, step-by-step structured tasks, and context-based learning have shown improvement in certain groups, the general picture suggests that the problem of representational transition is resistant to change and has not been fully resolved. This persistence suggests the issue lies not only in instruction but in meaning construction itself. The global difficulty with the epsilon-delta definition reflects a representational transition problem: transforming intuitive and visual signs into abstract, logically quantified symbols. Over time, the literature has shifted from simply describing errors to understanding the internal processes of learning, making a semiotic framework relevant for explaining how meaning is constructed.

The transformation of signs in describing the concept of a limit in calculus, especially in the context of epsilon-delta, can be challenging for students. This process requires a deep understanding of the interaction between the representamen, the object, and the interpretant; a relationship outlined in Peirce's triadic model [19]. This model suggests that one's understanding of a sign depends not only on the individual's interpretation, but also on the social context and accompanying prior experiences [20]. From this perspective, difficulties with epsilon-delta limits do not reflect merely procedural weakness but disruptions in semiotic transitions. Coordinating symbols, quantifiers, graphs, and verbal explanations reveal instability in the mediation between signs and mathematical objects, as described in Peirce's triadic semiotic theory. Without sufficient attention to this quantitative structure, many students have difficulty in explaining the formal definition of the limit and applying it in different situations [21].

Few studies have mapped the internal mechanisms of representational transition. While several studies have identified patterns of student difficulties in understanding limits and evaluated the effectiveness of specific teaching interventions [15]–[18], they generally focus on diagnosing errors or assessing method outcomes rather than providing a detailed mapping of the internal mechanisms of meaning construction. Relatively few studies have employed Peirce's triadic semiotic analysis [21] to trace the step-by-step transformation of representation, examining how representamen (signs such as graphs, language, and symbols), mathematical objects (concepts of limits), and interpretants (constructed meanings) interact and shift during learning. This gap highlights the need for research that not only identifies what goes wrong, but also uncovers how and why obstacles arise at each stage of the semiotic transition.

Understanding students' semiotic processes in constructing formal definitions of limits can be the basis for designing explicit teaching strategies that facilitate the transition of representation (from visual to symbolic to formal), thereby improving the ability to prove and sustainable conceptual understanding. In response to these problems and gaps, this study proposes a Peirce-based analytical approach to reveal the processes of constructing the concept of limits in students. These semiotic transition stages are identified using Peirce's triadic framework. Thus, this study attempts to reveal the semiotic processes that students go through when the construct formal definitions of limits according to Peirce's triadic framework, namely the semiotic processes that students go through when building an understanding of the formal definition of limits in the form of epsilon-delta.

Even though there are many literatures describe students' difficulties in understanding limits and evaluating the effectiveness of interventions, this study fills an important gap of previous studies. At this time of research, there is no other research discussing a more detailed mapping of the internal mechanisms of representational transitions interacting step by step when students shift from visual intuition to the formal epsilon-delta definition. The novelty of this research lies in the analytical application of Peirce's triadic semiotic analysis to trace each stage of the representational transformation. This study not only assesses whether an intervention improves performance, but also uncovers how and why obstacles arise at each stage of the semiotic transition, thus providing a foundation for designing teaching strategies explicitly facilitating the transfer from visual to symbolic and formal proof representations. Thus, this study tries to answer the question "How is the semiotic process of students in constructing the concept of a limit formally using Peirce's triadic semiotic framework?"

2. METHOD

2.1. Research design

This study employed a qualitative approach using a case study method to explore the semiotic process of students in constructing the formal concept of limits. Through in-depth analysis of the symbolic, verbal, and graphical representations used by students, the research revealed how conceptual understanding is formed through interaction with formal mathematical notation, particularly in expressions such as $\lim_{x \to c} f(x) = L$. The result offers important insights for developing more effective instructional approaches in calculus.

2.2. Participants

This study involved 21 third-semester undergraduate students of Mathematics Education at Universitas Islam Negeri Sayyid Ali Rahmatullah Tulungagung who had completed calculus courses and were still consolidating their formal understanding of limits. They completed a test consisting of 2 questions about the formal definition of limits, as shown in Figure 1. The test was conducted individually at the same time. Their answers were categorized based on: i) completeness of responses, ii) consistency of responses, iii) appearance of semiotic elements, and iv) comprehensive semiotic application, such as providing holistic explanations of the meanings behind the symbols. Then, participants who demonstrated strong written and oral communication skills were selected as interview subjects to explore the semiotic processes of how the students construct the formal concept of limits.

Answer the following questions.

1. Explain the formal definition of limit which states that "the limit $f(x)$ is L, if x it approaches c"! The explanation in the form of sentences and visual illustrations.
2. Prove the truth of the following limit using the formal definition of limit.
   a. $\lim_{x \to 1} 3x + 1 = 4$
   b. $\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2} = 5$

Figure 1. Test sheet on the formal definition of limit

2.3. Instruments

The research instrument was the researchers themselves with a test sheet and an interview guide. The test sheet contained two questions related to formal definition of limits, meanwhile the interview guide, which included key questions, was designed to dig out more information after administering the written test. The test sheet served as a tool to assess students' ability to formally explore the concept of limits. Meanwhile, the interview guide helped the researcher guide the discussion and ensure that all important aspects related to the students' semiotic processes in formally constructing the concept of limits could be explored. The written test sheet was used to capture the students' construction processes of the formal definition of limits, as shown in Figure 1.

The questions in the Figure 1 were designed to uncover the semiotic processes experienced by students in constructing the formal concept of limits. Question 1 was used to examine students' ability to write limit notation, the formal definition of limits, and to explain the formal definition using symbolic, visual, or verbal representations. Question 2 was used to examine students' ability to prove a limit using the formal definition.

The two questions were validated by two expert validators (Mathematics and Mathematics Education) to ensure alignment with the formal epsilon-delta definition and to elicit conceptual rather than procedural reasoning. Their open-ended structure enabled observation of students' semiotic processes by capturing the interaction between representamen (signs), mathematical objects, and constructed meanings. They were used to measure the presence of indicators listed in Table 1.

Then, to ensure that all important aspects related to the semiotic process of students in constructing the concept of limit formally can be explored, the researcher used several key questions as follows:

- Read the formal definition of the limit that you wrote!
- What do you understand about the notation $\lim_{x\to c} f(x) = L$? Explain!
- What do you understand about the symbols ε, δ, $|\quad|$ (absolute value), $<$, $>$, $\to$, $\Leftrightarrow$, $\forall$, $\exists$, $\ni$, $|x-c|$, $0<|x-c|$, $|x-c|<\delta$, $0<|x-c|<\delta$, $|f(x)-L|$, $|f(x)-L|<\varepsilon$, $0<|x-c|<\delta \Longrightarrow |f(x)-L|<\varepsilon$? Explain!
- What do you understand about the symbol $0<|x-c|<\delta \Longrightarrow |f(x)-L|<\varepsilon$? Explain!
- Explain the sketch of the graph that you drew!
- What meaning can you conclude from the formal definition of a limit, both analytically and graphically?
- Explain the process of proving the limit using the formal definition!

This study examined the semiotic processes according to Peirce's triadic framework, namely the semiotic processes that students go through when constructing an understanding of the formal definition of the limit in the form of epsilon-delta. The semiotic process was examined based on written tests and interviews as well as the semiotic indicators in Table 1.

Table 1. Research indicators based on Peirce's triadic theory

Semiotic elements	Indicator
Representamen	Students can i) write and show limit notation; ii) write and read (according to meaning) the formal definition of limit; iii) draw a graph sketch and explain the illustration of the formal definition of limit.
Object	Students can i) explain limit notation; ii) explain the epsilon-delta relationship; iii) explain the absolute value symbol, logical operators, quantifiers, crucial differences in the key symbols of the formal definition of limit; iv) explain logical implications; v) explain the formal definition of limit based on pictorial illustrations; vi) explain that limits are related to the behavior of functions around a point.
Interpretant	Students can i) explain the meaning (interpret) of the formal definition of limit analytically and graphically; ii) apply the formal definition of limit to prove the truth of the limit in cases where the point is defined or undefined; and iii) explain the processes of proving limits using the formal definition.

2.4. Data collection procedures

The data were collected through written tests and in-depth interviews [22]. The written tests were administered individually at the same time. The students completed a written test on the formal definition of limits for 120 minutes, which was conducted in the laboratory. Their written test answers were categorized based on completeness, consistency of answers, and the emergence of Peirce's triadic semiotic elements. Next, two students who were able to communicate well in writing and orally were selected as interview subjects to explore the students' semiotic processes in constructing the concept of limit formally. The interview was conducted in one session, lasting 90 minutes per subject, and managed in the interview room. Since this study explored the semiotic processes of students who demonstrated comprehensive triadic ability (applying semiotics comprehensively) in constructing the formal concept of limits, the selection of interview subjects was based on the following criteria: i) complete answers; ii) consistent answers; iii) complete Peirce's triadic semiotic elements; and iv) comprehensive implementation of semiotics.

Prior to data collection, participants were informed about the study's purpose and procedures and provided voluntary informed consent. They were free to withdraw at any time without consequences. All identifying information was anonymized using codes and the data were securely stored as well as used solely for research purposes.

2.5. Data analysis

The data analysis began by transcribing the interview results to facilitate further analysis. This transcription served as the basis for the researcher to code the data. The coding techniques were used to identify the main themes that emerged from the written test and interviews. Next, thematic analysis was done [23]–[25] by grouping the data according to the themes that emerged based on the indicators in Table 1. The analyzed data were the semiotic processes that students underwent while constructing their understanding of the formal limit definition in the epsilon-delta form.

The coding process in this study begins by assigning codes to excerpts from interviews and written tests related to Peirce's triadic semiotic theory (see Table 1). These codes were then grouped into major themes and analyzed to illustrate how students constructed the formal concept of limits. The analysis linked these themes to Peirce's triadic semiotic theory, revealing patterns in the semiotic processes students undergo in understanding the definition of limits.

2.6. Validity

The researcher used data triangulation [26] to ensure the accuracy of the findings by comparing the results of written tests and interviews, so that the findings obtained could be guaranteed to be consistent and reliable. In addition, member checking [26] was also done with subjects to confirm the findings. This process aimed to obtain input from subjects and ensured that the interpretations made by the researchers were in line with the students' experiences.

3. RESULTS AND DISCUSSION

Based on the written test results, 14 students were able to complete the questions completely, while the remaining 7 students were incomplete. Only 7 students answered completely and correctly in all aspects and consistently; only 6 students brought up semiotic elements; and only 2 students (DAZ and RNW) consistently demonstrated a comprehensive implementation of semiotics, meaning they not only answered technically correctly but also wrote a holistic explanation of the meaning behind the symbols. Thus, these two students were selected as interview subjects.

3.1. Results

Based on the results of a written test on 21 students, 2 students were selected as interview subjects. The results of the written test and in-depth interviews regarding the students' semiotic processes in constructing a formal definition of limit can be visualized as in Figure 2. The semiotic processes (semiosis) of formal construction of the concept of limit by students with a comprehensive triadic understanding in Figure 2 is explained as follows.

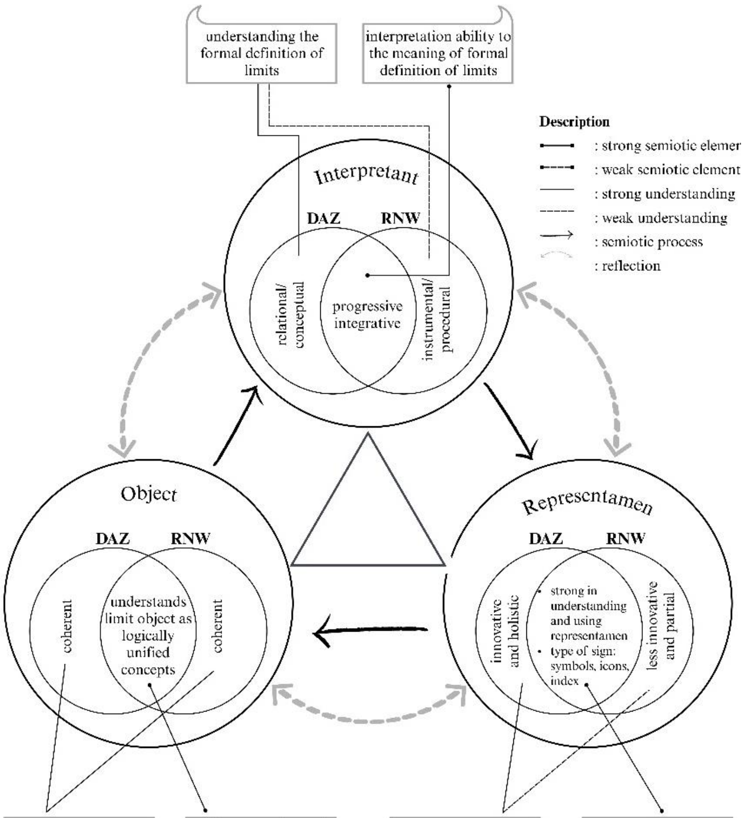

Figure 2. Formal construction processes of the limit concept

3.1.1. Semiosis at the representamen stage

Subject DAZ begins his understanding of the concept of limit by writing down the proper notation, $\lim_{x\to c} f(x) = L$, then includes the formal definition $\varepsilon - \delta$: for every, $\varepsilon > 0$ there exists $\delta > 0$ such that if, $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$, as shown in Figure 3. He explains that this shows that, $f(x)$ can be brought as close as desired to L when x is close enough to c without being equal to c, and then translates the definition into a simple sentence.

DAZ then creates a colored sketch showing the coordinate axes, the curve $y = f(x)$, points c on the x-axis and L on the y-axis, small hollow dots for undefined values and filled dots for values separated from the limit path, as shown in Figure 4. DAZ marks the vertical distance ε with blue curly brackets and lines $L + \varepsilon$ and $L - \varepsilon$, draws a dashed horizontal line to the curve and then a dashed vertical line to the axes x to define $c \pm \delta$, marks the distance δ on the x-axis as well as the points x in $(c - \delta, c)$ and $(c, c + \delta)$, and the values $f(x)$ in $(L - \varepsilon, L)$ and in pink and connects them with $(L, L + \varepsilon)$ pink dashed lines, and emphasizes that the condition on x must be met before $|f(x) - L| < \varepsilon$. Overall, DAZ's graphical-symbolic representation not only states the formal definition but also strengthens intuitive understanding of the limit concept.

Similar to DAZ, subject RNW also begins to write down the notation $\lim_{x\to c} f(x) = L$ and the definition of $\varepsilon - \delta$, then translated it into his own words to show that he truly understood that the value of $f(x)$ can be made as close as possible to L as long as x it is close to c (but not necessarily equal to c); this notation is presented in Figure 5.

$$\lim_{x\to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Figure 3. Formal notation and definition of limit written by DAZ

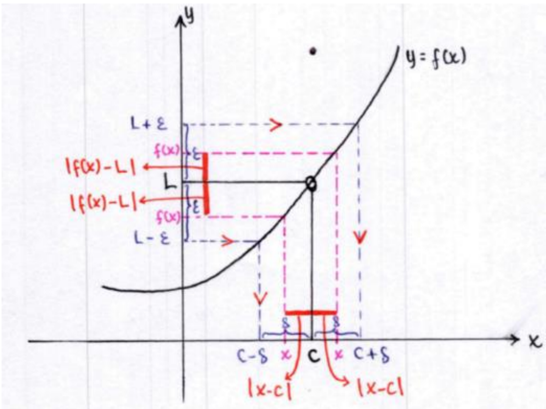

Figure 4. Sketch of the graph of the formal definition of the limit drawn by DAZ

(Formal definition of limit)

(Notation)

$$\lim_{x\to c} F(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x - c| < \delta, \Rightarrow |F(x) - L| < \varepsilon$$

notasi

Definisi Formal Limit

Figure 5. Formal notation and definition of limit written by RNW

Next, RNW makes a colored sketch, as shown in Figure 6, starting from the x-axis and the y-axis, draws the curve $y = f(x)$, marks the approach points c on the x-axis and the limit values L on the y-axis, and connects them with dashed lines for reference. RNW uses a hollow circle to indicate $f(c)$ undefined and a separate filled circle to indicate function values that are defined but differ from the limit path. He marks vertical intervals $L \pm \varepsilon$ with curly brackets and a dashed horizontal line to represent tolerance ε, then draws a vertical line from the limit intersection ε to the x-axis to define $c \pm \delta$ and notate δ the interval. The x points that satisfy $0 < |x - c| < \delta$ are marked and paired with the values $f(x)$ inside $(L - \varepsilon, L + \varepsilon)$ by a red dashed line connecting x to $f(x)$. RNW did forget to add arrows to these lines and acknowledged the omission in the interview excerpt, explaining that the selection x should have been indicated first. Overall, RNW's visualization not only writes the formal definition but also transforms it into an intuitive representation through colors, symbols, and graphical elements.

Representamen as a sign form that represents the concept of formal definition of limit shows that both DAZ and RNW have a strong and comprehensive understanding of this element, where both can write limit notation correctly, read and articulate formal definitions accurately, and illustrate concepts through creative and visual colorful graphics, so that the representamen serves as an effective symbolic mediator for both, not just mechanical memorization but a dynamic tool to convey mathematical ideas with profound clarity. However, a striking difference is seen in the more innovative and holistic DAZ, where the representamen becomes a dynamic visual-verbal tool with graphic illustrations that are both conceptually and aesthetically accurate, reducing reliance on memorization and achieving a high cognitive level for further exploration, while RNW tends to be more basic but still stable, focusing on clear reproduction such as precise notation and visual graphics without a strong emphasis on innovation or aesthetic appeal, indicating good initial abilities but still in the stage of recognizing signs.

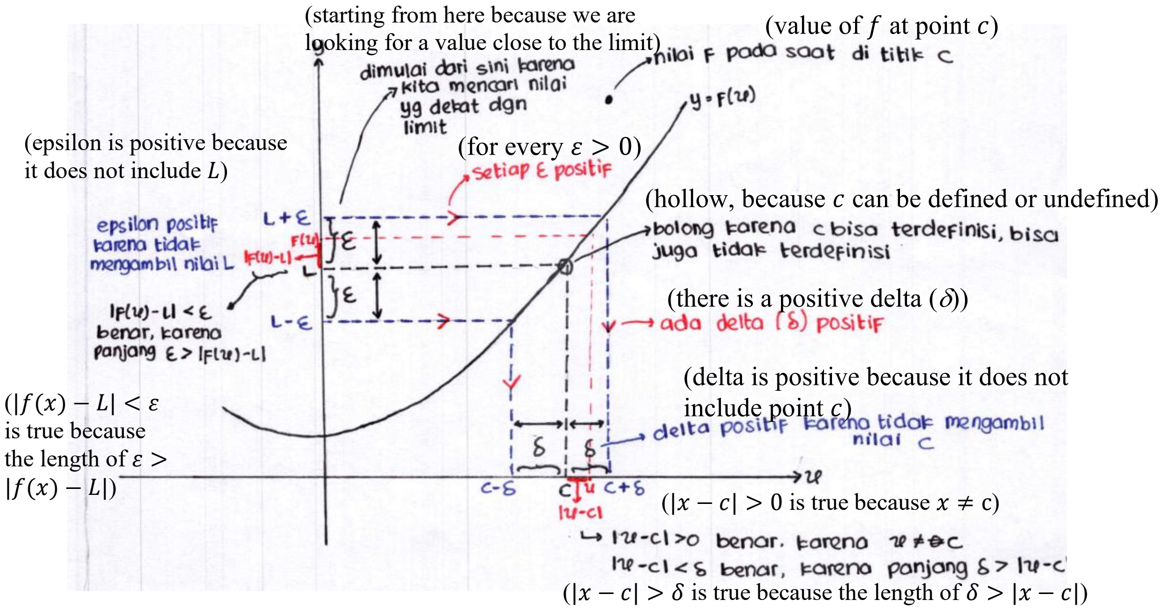

Figure 6. Sketch of the graph of the formal definition of the limit drawn by RNW

3.1.2. Semiosis at the object stage

The DAZ subject systematically presents the formal definition of the limit of a function by starting from the notation (see Figure 3) accompanied by graphical illustrations (see Figure 4) to clarify its geometric meaning. He explains it ε as the tolerance limit of the distance of values $f(x)$ to L and δ as the limit of the distance x to c, emphasizing the principle of "for every $\varepsilon > 0$ there is $\delta > 0$" as the core of the definition, and describes symbolic components such as absolute value, logical operators, and universal/existential quantifiers. DAZ can distinguish precisely between $|x - c|$ and $0 < |x - c|$, between $|x - c| < \delta$ and $0 < |x - c| < \delta$ to emphasize the exclusion of the point c, can also explain the conceptual difference between $|f(x) - L|$ and $|f(x) - L| < \varepsilon$, and then shows the logical implications $0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon$ through sketches and examples such as holes or discontinuities to emphasize that the limit refers to the behavior of the function as it x approaches c (see Figure 4). Overall, the explanation indicates a comprehensive, in-depth, and integrated understanding of DAZ between formal, intuitive, and visual aspects.

Next, RNW explains the formal definition of limit systematically and step by step with the help of graphical illustrations (see Figure 6), transforming the formal notation $\varepsilon - \delta$ into everyday explanations: ε as a tolerance on the value of a function and δ as a distance on the variables x that guarantees tolerance, and emphasizing the principle of "for every $\varepsilon > 0$ there is a $\delta > 0$". He also explains the meaning of absolute value, logical operators, and universal/existential quantifiers, can distinguish $|x - c|$ with $0 < |x - c|$ and $|x - c| < \delta$ with $0 < |x - c| < \delta$, can explain the difference between $|f(x) - L|$ and $|f(x) - L| < \varepsilon$, and finally summarizes the essence of the definition of $0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon$. RNW also emphasizes that $f(c)$ it is not necessarily defined or equal to L, so that the entire explanation is cohesive, visual, and makes it easier to understand the concept of limit.

Analysis of objects as realities represented by signs, namely the concrete essence of limit definitions such as the relationship between ε and δ, absolute value, quantifiers, and logical implications $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$. In this element, both subjects have a deep and coherent understanding of the core elements, including explanations of limit notation, relationships $\varepsilon - \delta$, absolute value, logical operators, quantifiers, crucial differences such as $|x - c|$ and $0 < |x - c|$, $|x - c| < \delta$ and $0 < |x - c| < \delta$, $|f(x) - L|$ and $|f(x) - L| < \varepsilon$, the logical implications, as well as the knowledge that functions do not have to be defined at the limit point are supported by pictorial illustrations, so that the limit object as an abstract concept has been represented in an integrated manner in the minds of both as a logical unity. The difference is that DAZ shows a more coherent integration by viewing objects as a whole, interrelated logical structure where elements support each other, resulting in a stable and holistic understanding with a thorough analysis of all aspects including complex logical implications, while RNW, although in-depth in explaining similar elements, emphasizes less on interconnectedness as a mutually supportive unity, focusing more on individual explanations that remain coherent and supported by illustrations.

3.1.3. Semiosis at the interpretant stage

Subject DAZ begins his explanation by referring back to the sketch in Figure 4 to emphasize the need for visual cues in understanding the abstract concept of limits, then states the core meaning of the formal definition: "as epsilon is chosen to be smaller around the limit value L, there will be smaller delta pairs around the point c so that the values x in the delta interval approach c and the values $f(x)$ approach L"; this understanding is confirmed through interview quotes. He places epsilon as the error tolerance on the function value and delta as the limit of the distance on the independent variable that guarantees that tolerance, and shows the reciprocal relationship "for every $\varepsilon > 0$ there is $\delta > 0$" as the basis for the formal proof.

DAZ then applies the definition to two cases, defined and undefined functions at the approximated points, by dividing the proof strategy into two stages: preliminary analysis and formal proof. In the case of $\lim_{x\to 1} 3x + 1 = 4$, DAZ quickly recognizes $f(1) = 4$, creates a sketch, as shown in Figure 7, to explain its graphical components, and then in the analysis stage reduces it $|(3x + 1) - 4| < \varepsilon$ to $|x - 1| < \frac{\varepsilon}{3}$ thus choosing $\delta = \frac{\varepsilon}{3}$ as the solution; in the proof stage he substitutes δ these and shows that $0 < |x - 1| < \frac{\varepsilon}{3}$ implies $|(3x + 1) - 4| < \varepsilon$, so the limit is proven. In addition, DAZ demonstrates conceptual flexibility by stating that many δ smaller choices, for example $\frac{\varepsilon}{7}, \frac{\varepsilon}{9}$ are also valid and proving each of them, which demonstrates a deep and integrated procedural, conceptual, and visual understanding of the definition of epsilon-delta.

In the case of the limit of a rational function$f(x) = \frac{2x^2-3x-2}{x-2}$ at the time $x \to 2$, DAZ correctly identifies that the function is undefined on $x = 2$ because the denominator becomes zero, and then factoring the numerator to obtain the equivalent form $2x + 1$ for all $x \neq 2$. From this simplified form we obtain $\lim_{x\to 2} 2x + 1 = 5$. To prove this result formally according to the definition of epsilon-delta, DAZ formulates the inequality $|(2x + 1) - 5| < \varepsilon$, which simplifies to $|2x - 4| = 2|x - 2| < \varepsilon$; thus, the election $\delta = \frac{\varepsilon}{2}$ guarantee that if $0 < |x - 2| < \delta$ then $\left|\left(\frac{2x^2-3x-2}{x-2}\right) - 5\right| < \varepsilon$, so that $\lim_{x\to 2} 2x + 1 = 5$ proven correct. DAZ also demonstrates the important mathematical understanding that the elimination of factors $(x-2)$ is only valid with conditions $x \neq 2$, because the definition of limit requires x close to but not equal to the limit point. The direct approach to the initial form (without prior simplification) provides consistency of results and shows flexibility in the choice of δ as positive constants that satisfy the same relation.

Furthermore, the subject RNW also explained by referring to the illustration in Figure 6 to emphasize the importance of visual clues in understanding the definition of $\varepsilon - \delta$. He explained that for every, $\varepsilon > 0$ there is $\delta > 0$ such that when x is in the interval approximately c $(c - \delta, c + \delta)$ then the distance $|f(x) - L|$ is less than ε, and in interviews he emphasized that no matter how small ε the chosen value, there is always a δ positive pair if x it is in that interval. His understanding of the reciprocal relationship $\varepsilon - \delta$ is considered profound: ε represents the tolerance on the function value and δ is the distance on the independent variable that ensures tolerance, which is the basis for formal proof of limits.

RNW also applies this definition to two cases, defined and undefined functions at the approximated point, by arranging the proof in two stages. For the case $\lim_{x\to 1} 3x + 1 = 4$, he was also able to deduce $f(1) = 4$, but failed to sketch the exact graph. RNW then performed a preliminary analysis by simplifying $|(3x + 1) - 4| < \varepsilon$ to $|x - 1| < \frac{\varepsilon}{3}$ such that choosing $\delta = \frac{\varepsilon}{3}$ (and showing that δ smaller ones can also be chosen), and in the formal proof stage he verified that the substitution $\delta = \frac{\varepsilon}{3}$ yields $3|x - 1| < \varepsilon$ an equivalent to $|(3x + 1) -$

$4| < \varepsilon$; in addition, RNW explored δ other options such as $\frac{\varepsilon}{4}$, $\frac{\varepsilon}{7}$, and $\frac{\varepsilon}{9}$ to demonstrate the non-unique nature of δ and flexibility in limit proofs.

In the second case analysis regarding the limit of a rational function $f(x) = \frac{2x^2-3x-2}{x-2}$ at $x \to 2$, RNW correctly states that the function is undefined at $x = 2$ and performs factorization to obtain an equivalent form $2x + 1$ for $x \neq 2$; then RNW derives $\left|\left(\frac{2x^2-3x-2}{x-2}\right) - 5\right| = |2x - 4| = 2|x - 2| < \varepsilon$ and chooses $\delta = \frac{\varepsilon}{2}$. However, although the proof $\varepsilon - \delta$ presented is algebraically correct, RNW ultimately shows that if $0 < |x - 2| < \frac{\varepsilon}{2}$ then the function value approaches 5 so that $\lim_{x\to 2} \frac{2x^2-3x-2}{x-2} = 5$, there is a conceptual weakness: RNW appears inconsistent in understanding the implications of the condition $x \neq 2$ on the operation of eliminating factors $(x - 2)$, as reflected in the interview excerpt, where the subject on the one hand states that elimination should not be done but in practice does it. This weakness indicates the need to strengthen the understanding of the legitimacy of algebraic manipulation in the context of limit definitions.

Interpretants who refer to the meaning generated through the interaction between representamen and objects, as applied in the processes of proving limits, show harmony in interpretative abilities that are progressive, namely developing gradually from a surface understanding to a deep conceptual understanding. This includes the ability to analyze and explain the meaning of formal definitions of limits analytically and graphically, apply limit proof methods to both defined and undefined points, and develop dynamic interpretants that reflect a deep understanding of limits as a process of logically approximating function values. This understanding implicitly strengthens Peirce's triadic connection in the context of a more complete and reflective conceptual understanding. The difference is that DAZ interpretants are holistic, applicable, and critical with the ability to describe the proof processes in detail, connecting theoretical elements into an integration for problem solving; while RNW is progressive but still developing, able to explain and apply but has difficulty illustrating graphical proofs and often forgets the prerequisites for the law of elimination in simplification, indicating a positive interpretant who needs strengthening visualization and procedures to achieve intuitive independence with high potential towards holistic.

Based on the research findings, DAZ and RNW exhibit similarities in the fundamental semiotic structure, yet differ in the quality of integration across each triadic element. To clarify these similarities and differences, Table 2 systematically summarizes their construction of representamen, object, and interpretant.

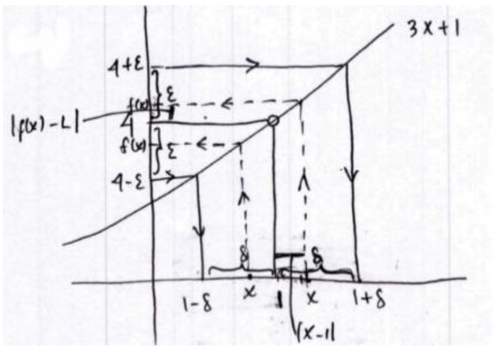

Figure 7. Graphic sketch $\lim_{x\to 1} 3x + 1 = 4$ drawn by DAZ

Table 2. Similarities and differences between DAZ and RNW in representamen, object, and interpretant

Semiotic elements	DAZ	RNW
Representamen	Strong in understanding and using representamen; innovative and holistic in producing representamen	Strong in understanding and using representamen; less innovative and more partial in producing representamen
Object	Understands the limit object as an integrated logical concept; coherent in presenting the object of the formal definition of the limit	Understands the limit object as an integrated logical concept; coherent in presenting the object of the formal definition of the limit
Interpretant	Interprets the meaning of the formal definition of limit in an integrative and progressive manner; demonstrates a relational/conceptual understanding	Interprets the meaning of the formal definition of limit in an integrative and progressive manner; demonstrates an instrumental/procedural understanding

3.2. Discussion

First, the representamen element. Both subjects demonstrated the use of rich, integrated representamen, and functioned as an effective symbolic mediator between the user (subject) and the formal concept of limit. The subjects used this representamen as a tool to interpret the meaning of symbols in the formal definition of limit. This is in accordance with previous research that representamen functions as a mediator in the triadic relationship between signs, objects, and their interpretations [27]–[31].

The findings showed that the representamen used include symbolic notation, detailed graphical representations (function graphs annotated with values around the approach points), and complementary written and oral verbal explanations. Theoretically, this is in accordance with Peirce's idea that signs (representamen) actively participate in meaning-making, functioning not only as signifiers but as dynamic elements that influence interpretation and understanding [19], [32]. Signs function as communication tools that enable individuals to understand and interpret the world around them [33]. Signs not only represent objects directly, but also function as a bridge between objects and interpretants [34], [35].

Based on the findings, the subjects used representations in the form of symbols, graphics (icons), and verbal representations to represent the concept of limits. This means that the subjects realized that symbols function as a very effective communication medium in conveying abstract ideas that are difficult to explain with ordinary words. In other words, the subjects were able to summarize complex concepts in a concise and clear form [36]–[38]. In relation to semiotics, symbols and images are used to represent mathematical concepts [39], symbols can be used to represent objects that only exist in the imagination [19]. Even previous research has stated that these symbols also help students to understand abstract concepts [40], [41], making learning mathematics more intuitive [42] and fun [43]. Mathematical symbols not only function as a communication tool, but also as a bridge that connects abstract thinking with a more concrete understanding [44], [45].

Research on the role of mathematical notation emphasizes that notation is not simply a concise tool, but shapes mathematical thinking and facilitates complex conceptual operations; thus, the appropriate use of formal notation by subjects supports their conceptual understanding of limits [46]. In addition, studies on the onto-semiotic approach, constructing meaning (knowledge) from signs, confirm that mastery of multiple representations (symbolic, graphic, and verbal/linguistic) enriches access to mathematical objects because different representations provide different perspectives on the same object [47]. In the context of meaning-making involving physical activity, the creative use of visuals (colored graphs, arrows, and annotations) by subjects also indicates greater affective-cognitive engagement and strengthens the link between sensory/visual experience and mathematical abstraction [44]. Thus, the representations constructed by subjects function not simply as formal memorization, but as dynamic tools that mediate thinking activities, shape symbolic understanding, and enable generalization to other limit problems; This is evidence that diverse and integrated representamen facilitates the construction of stable formal meaning [32], [46], [47].

Second, the object element. In the triadic perspective, an object refers to an abstract concept marked by a representamen. In this case, the mathematical object is the formal definition of the limit, the behavior of the function around the approach point, and the quantitative relations represented by these symbols. The findings showed that subjects not only write and read the formal definition, but can relate it to the relevant properties of the function, for example, the dependence of the distance $|x - c|$ on $|f(x) - L|$, as well as the quantitative conditions for choosing δ each ε. This shows that mathematical objects are not merely abstract labels, but are understood in their operational aspects, the way they appear in the processes of proof and problem solving. The concept of limit emphasizes that limit objects are not simply the result of mathematical calculations, but rather a cognitive experience that is systematically structured and manifested through graphical and analytical representations. Comprehensive understanding requires the ability to observe and analyze the behavior of a function as it approaches a certain point, and translate these observations into pairs of mathematically valid and consistent values [48]. Furthermore, studies of the semiotic processes of mathematics show that mathematical objects are often the result of an iterative process of interpretation between the representamen and the interpretant; in this case, objects are structured coherently because diverse representamen allow them to examine and confirm the properties of the object from many angles [46], [47]. In terms of object understanding, subjects in this category show integration between understanding the formal definition of limits (definition $\varepsilon - \delta$), operationalization abilities in tasks, and meta-cognitive awareness of the object's role in the proof, which is a characteristic of mature triadic understanding [48].

Third, the interpretant element. Interpretant in this category emerge as an intellectual and pragmatic consequence of the semiotic processes. The two subjects produced coherent, testable, and productive interpretants; they did not merely passively comprehend definitions, but they had capacity to draw new conclusions based on a deep understanding of the structure and behavior of the concept, formulated simple proofs, and predicted functional behavior in a limiting context. These findings demonstrated these interpretants in the form of consistent verbal explanations, selection strategies δ for ε certain items, and reflections on the usefulness of formal definitions in different situations. Theoretically, interpretants, according to Peirce, are not merely passive interpretations of signs, but rather cognitive effects produced by signs in the minds of interpreters (subjects). These interpretants constitute new rules of thought that open up space for meaningful action and are based on deep understanding [32]. Previous research has revealed that strong interpretants are related to transferability, can apply limiting definitions to a variety of problems and situations, and are usually facilitated by active interactions between verbal, symbolic, and visual representations [44], [47]. Meanwhile, in this study, the subjects were able to apply the formal definition of limits; however, they experienced difficulties in explaining the algebraic manipulation process due to forgetting the properties of absolute values. Interpretant also mediate the relationship between signs and their objects, enabling understanding and meaning-making [49]. Furthermore, literature examining the role of notation suggests that effective interpretant utilize notational structures to perform mental operations that accelerate derivation and proof; subjects' use of efficient notation appears to strengthen their interpretants, making them operational and reproducible on new tasks [46]. Therefore, interpretant in this category reflect a productive internalization of the formal concept of limit, an indicator of comprehensive and effective triadic understanding [32].

Overall, both subjects reflect a comprehensive triadic Semiosis: rich and interconnected representamen (symbols, graphics, and verbal/language), objects are understood in formal and operational aspects, and the resulting interpretants are productive and testable. This triadic integration shows empirical evidence that effective learning of the concept of limits occurs when students are given space to interact with various representations and are encouraged to explore mathematical objects through reasoning activities that encourage the emergence of new interpretants [47].

In this context, the approach taken by the first subject (DAZ), which employed creative and integrated semiotic representations, fostered dynamic and exploratory engagement, enabling students to navigate and internalize complex mathematical concepts [21], [50], [51]. In contrast, more traditional strategies, such as those of the second subject (RNW), emphasizing clear and explicit relationships between representamen and mathematical objects, supporting foundational understanding but potentially limiting creativity and flexible thinking [21], [52]. Although the first subject's methodology demonstrated potential for deeper conceptual understanding, debated remain regarding its practical implementation and measurable effectiveness compared to conventional methods [53], [54]. These findings underscore the importance of further research to evaluate how different semiotic strategies affect learning outcomes and suggest that a balanced integration of innovative and structured approaches can optimize both engagement and mastery of mathematical concepts.

This study has several limitations, particularly related to the sample size and scope of participants, which only involved students from a single university. The relatively small and homogeneous sample may restrict the representativeness of the data and limit the generalizability of the findings to a broader and more diverse student population. Consequently, the results should be interpreted with caution, as they may not fully capture variations in students' characteristics, learning experiences, or institutional contexts beyond the setting of this study.

4. CONCLUSION

Based on the results of the analysis of the formal construction processes of the concept of limit, it shows that the semiotic processes of students with a comprehensive triadic understanding occur in a triadic and comprehensive manner, including representamen, objects, and interpretants; which are mutually integrated and coherent. Rich and multimodal representamens act as the main mediator that enables the formation of a formal understanding of limits within the epsilon-delta framework. The mathematical objects are understood not merely as abstract labels, but through operational actions, namely patterns of functional behavior around a point. Interpretants emerge as productive cognitive effects in the form of solution strategies, the ability to prove, and the capacity to transfer ideas to other contexts. In other words, the integration of various representamen facilitates the transition from intuitive understanding to a stable formal understanding that can be stated with rigor through proof. These findings highlight the importance of designing instructional strategies that explicitly structure representational transitions. One possible approach is the use of structured tasks that require students to move between graphical representations and symbolic notation, and subsequently toward the formulation of the formal epsilon-delta proof. Repeated and reflective translation across representations (graphical⇔verbal⇔symbolic⇔formal) can strengthen the construction of interpretants and enhance students' logical consistency.

This study has limitations in the sample size and only involves students from one university. This small and homogeneous sample limits the ability to generalize the research findings to a broader student population. Therefore, further research is needed with a larger and more diverse sample from various universities to obtain a more representative picture of students' understanding of the concept of limits.

FUNDING INFORMATION

The authors state no funding involved.

AUTHOR CONTRIBUTIONS STATEMENT

This journal uses the Contributor Roles Taxonomy (CRediT) to recognize individual author contributions, reduce authorship disputes, and facilitate collaboration.

Name of Author	C	M	So	Va	Fo	I	R	D	O	E	Vi	Su	P	Fu
Beni Asyhar	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓		✓	
Purwanto		✓		✓	✓				✓	✓		✓		
Rustanto Rahardi		✓		✓	✓				✓	✓		✓		
Sudirman		✓		✓	✓				✓	✓		✓		

C : **C**onceptualization
M : **M**ethodology
So : **So**ftware
Va : **Va**lidation
Fo : **Fo**rmal analysis
I : **I**nvestigation
R : **R**esources
D : **D**ata Curation
O : Writing - **O**riginal Draft
E : Writing - Review & **E**diting
Vi : **Vi**sualization
Su : **Su**pervision
P : **P**roject administration
Fu : **Fu**nding acquisition

CONFLICT OF INTEREST STATEMENT

Authors state no conflict of interest.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author, [RR], upon reasonable request.

REFERENCES

[1] Kado, “Investigation into grade XI students’ misconception about the limit concept: a case study at Samtse Higher Secondary School in Bhutan,” *Asian Journal of Education and Social Studies*, vol. 15, no. 3, pp. 24–32, Mar. 2021, doi: 10.9734/AJESS/2021/V15I330381.

[2] Nurdin, S. F. Assagaf, and F. Arwadi, “Students’ understanding on formal definition of limit,” *Journal of Physics: Conference Series*, vol. 1752, no. 1, pp. 012082, Feb. 2021, doi: 10.1088/1742-6596/1752/1/012082.

[3] R. Puspitasari, M. A. Basir, and M. Aminudin, “Analysis of students’ algebraic reasoning level in learning limit function,” *Kontinu: Jurnal Penelitian Didaktik Matematika*, vol. 8, no. 1, pp. 47–65, May 2024, doi: 10.30659/KONTINU.8.1.47-65.

[4] R. Oktaviyanthi, R. N. Agus, M. L. B. Garcia, and K. Lertdechapat, “Cognitive load scale in learning formal definition of limit: a rasch model approach,” *Infinity Journal*, vol. 13, no. 1, pp. 99–118, Jan. 2024, doi: 10.22460/INFINITY.V13I1.P99-118.

[5] M. Slavíčková and M. Slavíčková, “Implementation of digital technologies into pre-service mathematics teacher preparation,” *Mathematics 2021, Vol. 9,* vol. 9, no. 12, Jun. 2021, doi: 10.3390/MATH9121319.

[6] I. N. Yenti, Y. S. Kusumah, and J. A. Dahlan, “Students’ perceptions about the implementation of peer-assisted reflection learning in multivariable calculus lectures,” *Ta’dib*, vol. 24, no. 1, pp. 9–19, Jun. 2021, doi: 10.31958/JT.V24I1.3315.

[7] W. Liu, K. A. Jamaludin, M. I. Hamzah, and Z. Song, “Implementation of technical communication pedagogy in higher education institutions: A systematic literature review,” *International Journal of Learning, Teaching and Educational Research*, vol. 23, no. 9, pp. 307–324, Sep. 2024, doi: 10.26803/IJLTER.23.9.16.

[8] N. A. Utami, C. Sa’dijah, and T. D. Chandra, “Students’ mathematical representation in solving exponential function,” *AKSIOMA: Jurnal Program Studi Pendidikan Matematika*, vol. 12, no. 1, pp. 450–458, Mar. 2023, doi: 10.24127/AJPM.V12I1.6166.

[9] A. N. L. Rahmawati, C. Sa’dijah, and L. Anwar, “High school students’ ability with translation among mathematical representations in solving the HOTS-based problems,” *Jurnal Pendidikan MIPA*, vol. 23, no. 4, pp. 1887–1899, 2022, doi: 10.23960/JPMIPA/V23I4.PP1887-1899.

[10] R. N. Yuliandari, C. Sa’dijah, Susiswo, and Purnomo, “Teachers’ multiple representations on teaching fractions at elementary school: A commognitive framework,” *AL-ISHLAH: Jurnal Pendidikan*, vol. 16, no. 2, pp. 1005–1018, Jun. 2024, doi: 10.35445/ALISHLAH.V16I2.5048.

[11] N. N. Alyani and Subanji, “Mathematical representation translations of junior high school students on quadratic function graph material,” *Math Educa Journal*, vol. 8, no. 2, pp. 154–170, Nov. 2024, doi: 10.15548/mej.v8i2.9814.

[12] H. B. Chand, B. C. Luitel, B. P. Pant, and S. Belbase, “Exploring undergraduate students’ understanding of the limit of a function,” *Academic Journal of Mathematics Education*, vol. 7, no. 1, pp. 14–28, Dec. 2024, doi: 10.3126/AJME.V7I1.81421.

[13] E. Rexigel, J. Kuhn, S. Becker, and S. Malone, “The more the better? A systematic review and meta-analysis of the benefits of more than two external representations in STEM education,” *Educational Psychology Review*, vol. 36, no. 4, pp. 1–59, Dec. 2024, doi: 10.1007/S10648-024-09958-Y.

[14] K. N. Mpuangnan, B. K. Adjei, and S. N. Govender, “Impact of multiple representations-based instruction on teaching and learning of linear equations,” *Journal of Pedagogical Sociology and Psychology*, vol. 6, no. 1, pp. 58–76, Mar. 2024, doi: 10.33902/JPSP.202425242.

[15] O. J. Evardo Jr and E. C. Itaas, "Enhancing students' performance on least taught topics in basic calculus through Moodle-based courseware package," *Journal of Mathematics and Science Teacher*, vol. 4, no. 2, pp. em060, Apr. 2024, doi: 10.29333/MATHSCITEACHER/14310.

[16] T. Díaz-Chang and E.-H. Arredondo, "Exploring the relationship between tacit models and mathematical infinity through history," *International Electronic Journal of Mathematics Education*, vol. 18, no. 2, pp. em0730, Apr. 2023, doi: 10.29333/IEJME/12823.

[17] T. S. Love, "Practitioner journals: an integral component for advancing research, knowledge, practices, and discussion in STEM education," *Journal of Technology Studies*, vol. 49, no. 1, pp. 1–7, May 2024, doi: 10.21061/JTS.414.

[18] J. M. Gavilán-Izquierdo and I. Gallego-Sánchez, "How an upper secondary school teacher provides resources for the transition to university: A case study," *International Electronic Journal of Mathematics Education*, vol. 16, no. 2, pp. em0634, May 2021, doi: 10.29333/IEJME/10892.

[19] D. Chandler, *Semiotics: the basics*. New York: Routledge, 2022.

[20] K. Moradi, A. Ghaebi, and M. K. A. Kamran, "Effective components on the process of concept and meaning creation and their application in ontologies: focusing on Peirce's triadic sign model," *The Electronic Library*, vol. 40, no. 4, pp. 486–502, Aug. 2022, doi: 10.1108/EL-10-2021-0185.

[21] R. Purwasih, Turmudi, J. A. Dahlan, E. Irawan, and S. Minasyan, "A semiotic perspective of mathematical activity: the case of integer," *Infinity Journal*, vol. 13, no. 1, pp. 271–284, Jan. 2024, doi: 10.22460/INFINITY.V13I1.P271-284.

[22] R. K. Yin, *Case study research and applications: design and methods*. 6 ed, Los Angeles: SAGE Publications, Inc., 2018.

[23] X. Gu *et al.*, "A qualitative thematic analysis exploring Chinese young adults' experiences in decision making on the management of low-risk papillary thyroid cancer," *Thyroid*, vol. 34, no. 12, pp. 1486–1494, Dec. 2024, doi: 10.1089/THY.2024.0210.

[24] L. Kogen, "Qualitative thematic analysis of transcripts in social change research: Reflections on common misconceptions and recommendations for reporting results," *International Journal of Qualitative Methods*, vol. 23, pp. 1–11, Jan. 2024, doi: 10.1177/16094069231225919.

[25] V. V. Lee, S. C. C. van der Lubbe, L. H. Goh, and J. M. Valderas, "Harnessing ChatGPT for thematic analysis: are we ready?" *Journal of Medical Internet Research*, vol. 26, no. 1, pp. 1–11, May 2024, doi: 10.2196/54974.

[26] J. Vella, "In pursuit of credibility: evaluating the divergence between member-checking and hermeneutic phenomenology," *Research in Social and Administrative Pharmacy*, vol. 20, no. 7, pp. 665–669, Jul. 2024, doi: 10.1016/J.SAPHARM.2024.04.001.

[27] P. Konderak, "Towards an integration of two aspects of semiosis – a cognitive semiotic perspective," *Sign Systems Studies*, vol. 49, no. 1–2, pp. 132–165, Jun. 2021, doi: 10.12697/SSS.2021.49.1-2.06.

[28] C. W. Suryaningrum, Purwanto, Subanji, H. Susanto, Y. D. W. K. Ningtyas, and M. Irfan, "Semiotic reasoning emerges in constructing properties of a rectangle: a study of adversity quotient," *Journal on Mathematics Education*, vol. 11, no. 1, pp. 95–110, Jan. 2020, doi: 10.22342/jme.11.1.9766.95-110.

[29] A. Olteanu, C. Campbell, and S. Feil, "Naturalizing models: new perspectives in a Peircean key," *Biosemiotics*, vol. 13, no. 2, pp. 179–197, Aug. 2020, doi: 10.1007/S12304-020-09385-W.

[30] P. Paraskevaidis dan A. Weidenfeld, "Perceived and projected authenticity of visitor attractions as signs: a Peircean semiotic analysis," *Journal of Destination Marketing & Management*, vol. 19, pp. 100515, Mar. 2021, doi: 10.1016/J.JDMM.2020.100515.

[31] H. Palayukan, Purwanto, Subanji, and Sisworo, "Student's semiotics in solving problems geometric diagram viewed from Peirce perspective," *AIP Conference Proceedings*, vol. 2215, no. 1, Apr. 2020, doi: 10.1063/5.0000719/615660.

[32] C. Barnham, "Hegel and the Peircean 'object'," *Sign Systems Studies*, vol. 48, no. 1, pp. 101–124, Jun. 2020, doi: 10.12697/SSS.2020.48.1.06.

[33] M. Danesi, *The quest for meaning: a guide to semiotic theory and practice*. 2 ed, Toronto: University of Toronto Press, 2020.

[34] E. Andrews, H. Bierman, B. Hannon, and H. Ling, "Semiosis and embodied cognition: the relevance of Peircean semiotics to cognitive neuroscience," *Sign Systems Studies*, vol. 52, no. 1–2, pp. 49–69, Sep. 2024, doi: 10.12697/SSS.2024.52.1-2.02.

[35] E. Ghibaudi, "An exercise of applied epistemology: Peirce's semiosis implemented in the representation of protein molecules," *Substantia*, vol. 8, no. 2, pp. 45–56, May 2024, doi: 10.36253/SUBSTANTIA-2558.

[36] L. Li, "Mathematical symbols in academic writing: the case of incorporating mathematical ideals in academic writing for education researchers," *Brock Education Journal*, vol. 33, no. 1, pp. 66–84, Feb. 2024, doi: 10.26522/BROCKED.V33I1.1122.

[37] A. Andriyani, M. Fitrianawati, I. A. Khalil, M. Barida, and R. C. I. Prahmana, "Stimulating mathematical communication with SPECOMATSO technology development based on digital literacy," *Jurnal Elemen*, vol. 9, no. 2, pp. 475–493, Jul. 2023, doi: 10.29408/JEL.V9I2.15869.

[38] H. Douglas, M. G. Headley, S. Hadden, and J. A. Lefevre, "Knowledge of mathematical symbols goes beyond numbers," *Journal of Numerical Cognition*, vol. 6, no. 3, pp. 322–354, Dec. 2020, doi: 10.5964/JNC.V6I3.293.

[39] N. Presmeg, L. Radford, W.-M. Roth, and G. Kadunz, *Semiotics in mathematics education*, ICME-13 Topical Surveys, Cham: Springer International Publishing, 2016, doi: 10.1007/978-3-319-31370-2.

[40] F. R. Dobler, M. R. Henningsen-Schomers, and F. Pulvermüller, "Verbal symbols support concrete but enable abstract concept formation: evidence from brain-constrained deep neural networks," *Language Learning*, vol. 74, no. S1, pp. 258–295, Jun. 2024, doi: 10.1111/LANG.12646.

[41] L. C. H. Venenciano, S. L. Yagi, and F. K. Zenigami, "The development of relational thinking: a study of measure up first-grade students' thinking and their symbolic understandings," *Educational Studies in Mathematics*, vol. 106, no. 3, pp. 413–428, Mar. 2021, doi: 10.1007/S10649-020-10014-Z.

[42] U. W. Hultdin and M. Norqvist, "Students' reception of two alternative arrangements of mathematical symbols and words: differences in focus and text navigation," *The Journal of Mathematical Behavior*, vol. 75, pp. 101159, Sep. 2024, doi: 10.1016/J.JMATHB.2024.101159.

[43] B. Roberts, C. MacLeod, and M. Fernandes, "Symbol superiority: why $ is better remembered than 'dollar'," *Journal of Vision*, vol. 23, no. 9, pp. 4575–4575, Aug. 2023, doi: 10.1167/JOV.23.9.4575.

[44] S. Chance, F. Fayyaz, A. L. Campbell, N. P. Pitterson, and S. Nawaz, "Guest editorial special issue on conceptual learning of mathematics-intensive concepts in engineering," *IEEE Transactions on Education*, vol. 67, no. 4, pp. 491–498, 2024, doi: 10.1109/TE.2024.3416649.

[45] G. Greenberg, "The iconic-symbolic spectrum," *The Philosophical Review*, vol. 132, no. 4, pp. 579–627, Oct. 2023, doi: 10.1215/00318108-10697558.

[46] C. Cellucci, "The role of notations in mathematics," *Philosophia*, vol. 48, no. 4, 2020, doi: 10.1007/S11406-019-00162-9.

[47] P. Beltrán-Pellicer and J. D. Godino, "An onto-semiotic approach to the analysis of the affective domain in mathematics education," *Cambridge Journal of Education*, vol. 50, no. 1, pp. 1–20, Jan. 2020, doi: 10.1080/0305764X.2019.1623175.

[48] M. Arnal-Palacián, "Infinite limit of a function at infinity and its phenomenology," *Annales Universitatis Paedagogicae Cracoviensis | Studia ad Didacticam Mathematicae Pertinentia*, vol. 14, pp. 25–41, Dec. 2022, doi: 10.24917/20809751.14.3.

[49] C. Paolucci, "A semiotic lifeworld. Semiotics and phenomenology: Peirce, Husserl, Heidegger, Deleuze, and Merleau-Ponty," *Semiotica*, vol. 2024, no. 260, pp. 25–43, Sep. 2024, doi: 10.1515/SEM-2024-0152.
[50] M. Z. G. Man, R. Hidayat, M. K. Kashmir, N. F. Suhaimi, M. Adnan, and A. Saswandila, "Design thinking in mathematics education for primary school: a systematic literature review," *Alifmatika: Jurnal Pendidikan dan Pembelajaran Matematika*, vol. 4, no. 1, pp. 17–36, May 2022, doi: 10.35316/ALIFMATIKA.2022.V4I1.17-36.
[51] B. Durmaz, "The use of literary elements in teaching mathematics: a bibliometric analysis," *Journal of Teacher Education and Lifelong Learning*, vol. 5, no. 1, pp. 152–172, Jun. 2023, doi: 10.51535/TELL.1232736.
[52] R. Hidayat, M. Adnan, M. F. N. L. Abdullah, and Safrudiannur, "A systematic literature review of measurement of mathematical modeling in mathematics education context," *Eurasia Journal of Mathematics, Science and Technology Education*, vol. 18, no. 5, pp. em2108, Apr. 2022, doi: 10.29333/EJMSTE/12007.
[53] C. W. Suryaningrum, Misyana, and T. E. Jatmikowati, "Playing mathematics in early childhood based on semiotics," *Jurnal Obsesi: Jurnal Pendidikan Anak Usia Dini*, vol. 6, no. 2, pp. 601–610, Jun. 2022, doi: 10.31004/OBSESI.V6I2.1341.
[54] J. Zhou, "Grounding without a ground: a Peircean conception of semiotic ethics," *Transactions of the Charles S. Peirce Society: A Quarterly Journal in American Philosophy*, vol. 61, no. 1, pp. 52–71, Dec. 2025, doi: 10.2979/CSP.00043.

BIOGRAPHIES OF AUTHORS

Beni Asyhar is a doctoral candidate in Mathematics Education at the Department of Mathematics, Universitas Negeri Malang and is also active as a lecturer in the Department of Mathematics Education, Universitas Islam Negeri Sayyid Ali Rahmatullah Tulungagung. He completed his undergraduate education in Mathematics and continued to the Masters level in Mathematics Education. His research focus covers various topics in the field of mathematics education, including the development of HOTS-based learning tools to improve students' visual thinking skills, and the development of a hybrid learning management system. He is also interested in the study of mathematical proof, student errors in solving problems, and mapping visual thinking profiles in solving mathematical problems. He can be contacted at email: beni.asyhar.2103119@students.um.ac.id.

Purwanto is a professor in the Department of Mathematics, Universitas Negeri Malang. He earned his bachelor's degree (Drs.) in Mathematics Education and earned his doctorate (Ph.D.) in Pure Mathematics. His areas of expertise and research are very broad, including mathematics education, mathematical literacy, students' and teachers' mathematical thinking, and labeling in graph theory. Several of his studies explore symbolic and visual representations, analogical reasoning, students' conceptual errors, mathematical communication, and the use of technology such as Augmented Reality in numeracy learning. With extensive contributions to mathematics education and research, he is one of the leading academics in his field. He can be contacted at email: purwanto.fmipa@um.ac.id.

Rustanto Rahardi is a lecturer in the Department of Mathematics, Universitas Negeri Malang. He earned his Drs. in Mathematics Education, M.Si. in Pure Mathematics, and earned his doctorate (Dr.) in Mathematics Education. His research focuses on various aspects of mathematical learning and thinking, such as symbolic-visual representation, critical and creative thinking skills, conceptual understanding, and students' learning and cognitive styles. He also explores many contemporary mathematics education issues such as the use of manipulative and digital media, the development of LKS based on innovative approaches (PBL, Guided Discovery, REACT), and the analysis of student errors based on Newman's theory. He can be contacted at email: rustanto.rahardi.fmipa@um.ac.id.

Sudirman is a lecturer in the Department of Mathematics, Universitas Negeri Malang. He completed his undergraduate education (Drs.) in Mathematics Education, master's degree (M.Si.) in Pure Mathematics, and earned a doctorate (Dr.) in Mathematics Education. His research focuses including students' mathematical literacy and thinking literacy, symbolic representation and mathematical communication processes, mathematical argumentation, and mathematical problem modeling and solving with innovative approaches such as Polya, STEM, and augmented reality. He also studies mathematical thinking based on personality, gender, and students' cognitive engagement in online learning. He can be contacted at email: sudirman.fmipa@um.ac.id.